Annals of Operations Research (2019) 282:27–57
https://doi.org/10.1007/s10479-018-2941-9

S.I.: APPLICATION OF O. R. TO FINANCIAL MARKETS

Closed-form variance swap prices under general affine GARCH models and their continuous-time limits

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Published online: 23 June 2018

Abstract

Fully explicit closed-form expressions are developed for the fair strike prices of discrete-time variance swaps under general affine GARCH type models that have been risk-neutralized with a family of variance dependent pricing kernels. The methodology relies on solving differential recursions for the coefficients of the joint cumulant generating function of the log price and the conditional variance processes. An alternative derivation is provided in the case of Gaussian innovations. Using standard assumptions on the asymptotic behavior of the GARCH parameters as the sampling frequency increases, the diffusion limit of a Gaussian GARCH model is derived and the convergence of the variance swap prices to its continuous-time limit is further investigated. Numerical examples on the term structure of the variance swap rates and on the convergence results are also presented.

Keywords Variance swaps · Realized variance · Affine GARCH models · Variance dependent pricing kernels · Diffusion limits

JEL Classification: C58 · G13 · G17

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1 Introduction

Volatility derivatives are financial instruments with *volatility* as the underlying tradeable. Here, the term "volatility" has a broad meaning and can refer to the realized volatility of the stock or index, the volatility index (VIX), the spot volatility, etc. Variance swaps (VS) are the first and most prominent volatility derivatives introduced in the market.[1] These are contracts under which the actual realized variance over a period is exchanged (or swapped) with a pre-specified variance level named the *fair strike*. Traditionally, practitioners carry out volatility trading (i.e. trade with a viewpoint towards the direction of volatility movement) through delta hedging options. The economic motivation for introducing the variance swaps into the market is to offer investors direct exposure to the volatility of the underlying asset (either index or individual name stock), without the need of options. Buying a variance swap is equivalent to holding a *long* position in the volatility at the strike level, which means that one makes a profit if the market delivers more volatility than the strike level, and vice versa. On the hedging side, from the above discussions, it is clear that the variance swap can be replicated by a static portfolio of options. These two features, namely the direct exposure to volatility, and the ease of replication, make variance swaps a useful and attractive financial instrument.

Volatility derivatives are in practice based on the realized variance, which is the sum of squares of the discretely sampled log stock prices. However, most academic literature concerns the price of continuously sampled contracts as approximations, that is, the realized variance (RV) is replaced by the quadratic variation (QV) process (see studies in Broadie and Jain (2008), Jarrow et al. (2013), Bernard and Cui (2014), Bernard et al. (2014), and Lian et al. (2014)). This approximation deteriorates as the sampling frequency decreases and it is hence of interest to obtain an exact formula for discretely sampled volatility derivatives. Under the assumption of continuous asset price paths, the QV process reduces to the familiar integrated variance (IV). There are two main approaches to VS pricing based on the QV methodology: the first approach, as derived in Demeterfi et al. (1999) and Carr and Madan (1998) consists in calculating the VS strike price by replicating the swap payoff with a portfolio of European options. This replication strategy requires the assumption of continuous price paths and is the basis for the VIX calculation. The second is based on calculating the expected QV of the asset process explicitly. This latter approach does not require the continuous asset price assumption, thus allowing for non-Gaussian asset pricing models. Notable examples include the pricing of VS in the Heston model, as derived in Brockhaus (2000), and a similar approach is applied to Nelson's GARCH diffusion model in Javaheri et al. (2004), and a GARCH-like model with delay in Swishchuk (2005). Variance swaps for a non-Gaussian Ornstein–Uhlenbeck stochastic volatility model are considered in Benth et al. (2007), affine jump diffusion and jump diffusion models with stochastic volatility in Broadie and Jain (2008), and for general affine models in Kallsen et al. (2011). Volatility swaps, on the other hand, cannot be replicated using European options, and closed-form solutions for the fair volatility swap strike price, in general, do not exist. One approach commonly seen in the literature in pricing volatility swaps is based on the approximation presented in Brockhaus (2000). Zhu and Lian (2011) consider variance and volatility swap pricing without relying on the RV approximation through the QV; an exact solution for the fair strike price for VS in the Heston model is derived and the effect of discretely sampled asset prices is examined.

[1] For a survey of the academic literature about volatility derivatives, please refer to Carr and Lee (2009), and references therein.

Since their introduction by the Chicago Board Options Exchange (CBOE) in 2004, VIX derivatives have gained a lot of interest in practice and in academic studies. Most of the literature so far has focused on the continuous time setting, under which, either the underlying asset price or the VIX itself is modelled by continuous time stochastic processes. For the valuation of VIX derivatives and related empirical work, please refer to the recent papers of Duan and Yeh (2010), Mencía and Sentana (2013), Song and Xiu (2016), and the references therein. In discrete-time, VIX index pricing has been considered under non-affine GARCH models by Hao and Zhang (2013) for Gaussian innovations and by Lalancette and Simonato (2017) for Johnson distributed noise. One of the main disadvantages of non-affine (GARCH) settings is that they do not generally allow for explicit solutions for VIX related quantities. Wang et al. (2017) obtained a closed-form pricing formula for the CBOE VIX futures under the Heston and Nandi (2000a) model driven by Gaussian innovations (HN-GARCH), while Chorro and Rahantamialisoa (2016) derived an analytic solution for the VIX index under an affine Inverse Gaussian GARCH model. The VIX formula in the affine GARCH models is given in terms of the square root of an affine function of the terminal variance process, which can be priced from knowing the moment generating function of the terminal variance process only. However, the discretely sampled variance swap has a path-dependent payoff that involves accumulating squares of log returns. Thus, it is more challenging to valuate discretely sampled VS compared to VIX derivatives under the affine GARCH models.

In a recent paper, Badescu et al. (2018) considered the pricing of variance swaps in general non-affine GARCH settings and provide closed-form expressions only for Gaussian environments. They provided explicit VS pricing estimates based on non-Gaussian GARCH assumptions for the underlying, and investigated the convergence of VS rates to continuous-time counterparts. To the best of the authors' knowledge, this represented the first attempt in the literature to price VS within a GARCH environment.[2]

In this paper, we consider the pricing of VS contracts within a general class of affine GARCH models. More specifically, we provide for the first time an explicit exact closed-form expression for the fair strike of discretely monitored VS and investigate the convergence of VS rates for a special affine Gaussian GARCH model.

The main contributions of the paper are illustrated in detail below. First, we consider a general affine GARCH pricing framework via a variance dependent pricing kernel and we derive a closed-form exact formula for the fair strike price of discretely sampled VS. In particular, we show that when the multi-step risk-neutral joint cumulant generating function (c.g.f) of log-price and variance processes is affine, we can express the variance swap price as a quadratic function of the spot variance with coefficients depending only on the first and second order derivatives of the one-step bivariate c.g.f. of log-price and variance. Furthermore, for the special HN-GARCH model, we provide an alternative derivation of the VS price using a direct evaluation methodology, which does not rely on the more general recursive approach based on the affine c.g.f. structure, but leads to the same result. Secondly, using standard asymptotic constraints on the GARCH parameters as in Nelson (1990), we derive the continuous weak diffusion limit of the HN-GARCH model under the proposed variance dependent pricing kernel. The continuously sampled fair strike of VS is computed based on this process and we further show that this rate represents the limit of the GARCH based counterpart constructed based on the RV definition. Finally, we provide two numerical exercises in which we illustrate the term structure of the VS rates implied by the historical asset returns on S&P 500 via a

[2] Note that Badescu et al. (2018) considered a non-affine GARCH setting while our current work is focused on affine GARCH models. Furthermore, the pricing methods employed and the convergence results are completely different.

rolling window estimation technique, and we numerically investigate the convergence from discretely to continuously sampled fair strikes for different maturities and levels of risk-neutral to physical variance ratio.

The rest of the paper is structured as follows: Section 2 introduces the affine GARCH setup and discusses the choice of risk-neutralization within the class of variance dependent pricing kernels. Section 3 presents the valuation formulas for discretely sampled variance swaps and discusses the HN-GARCH special case. Section 4 introduces the weak diffusion limit of the underlying HN-GARCH model based on the variance dependent pricing kernel and demonstrates theoretically the convergence of the discrete VS prices to their continuous counterparts. Section 5 presents numerical examples illustrating the convergence. Section 6 concludes the paper with future research directions.

2 Affine GARCH models and exponential pricing kernels

In this section, we introduce a general derivatives pricing framework within the class of affine GARCH models, and we later discuss in detail a prominent special case, the HN-GARCH model.

2.1 A general affine GARCH pricing framework

We consider a discrete-time setting defined on the interval $[0, \ldots, T]$, which consists of nT trading dates taking place at equally spaced subintervals of length $\Delta = 1/n$. The filtered probability space associated to this framework is denoted by $\left(\Omega, \mathcal{F}, \{\mathcal{F}_{k\Delta}\}_{k=0,\ldots,nT}, P\right)$, where P is the underlying physical measure. We let $Y = \{Y_{k\Delta}\}_{k=0,\ldots,nT} := \{\log S_{k\Delta}\}_{k=0,\ldots,nT}$ be the log-asset price process at time $k\Delta$, and we denote by $y_{k\Delta} := Y_{k\Delta} - Y_{(k-1)\Delta}$ the log-return process over the period $[(k-1)\Delta, k\Delta]$. We assume that the returns dynamics are driven by a single factor, which is taken here to be the conditional variance process $h = \{h_{k\Delta}\}_{k=1,\ldots,nT}$, an $\mathcal{F}_{k\Delta}$-predictable process defined by $h_{k\Delta} := \mathrm{Var}^P\left[y_{k\Delta} \mid \mathcal{F}_{(k-1)\Delta}\right]$.

In general, the discretized version of an affine GARCH(1,1) model dynamic can be summarized by the following equations:

$$y_{k\Delta} = f_1\left(h_{k\Delta}, \theta(\Delta)\right) + \sqrt{\Delta}\sqrt{h_{k\Delta}}\epsilon_{k\Delta}, \tag{2.1}$$

$$h_{(k+1)\Delta} = f_2\left(h_{k\Delta}, \theta(\Delta)\right) + f_3\left(h_{k\Delta}, \epsilon_{k\Delta}, \theta(\Delta)\right). \tag{2.2}$$

Here $\epsilon_{k\Delta}$ is a sequence of $\mathcal{F}_{(k-1)\Delta}$-conditional i.i.d. random variables with a finite moment generating function and θ is a vector of parameters that obey some constraints in order to ensure the non-negativity and stationarity of the conditional variance process.[3] The functions f_1 and f_2 are affine in $h_{k\Delta}$, while the form of the news function f_3 is dictated by the innovation distribution. More specifically, it is chosen such that the conditional bivariate cumulant generating function (c.g.f.) of $\left(y_{k\Delta}, h_{(k+1)\Delta}\right)$ has the following affine structure:

$$\begin{aligned} C_{\left(y_{k\Delta}, h_{(k+1)\Delta}\right)}\left(\phi, \psi \mid \mathcal{F}_{(k-1)\Delta}\right) &= \log \mathrm{E}^P\left[\exp\left(\phi y_{k\Delta} + \psi h_{(k+1)\Delta}\right) \mid \mathcal{F}_{(k-1)\Delta}\right] \\ &= A\left(\phi, \psi; (k-1)\Delta, k\Delta\right) + B\left(\phi, \psi; (k-1)\Delta, k\Delta\right) h_{k\Delta}. \end{aligned} \tag{2.3}$$

[3] Specific constraints will be provided for the Heston and Nandi (2000a) GARCH(1,1) model introduced in Sect. 2.2.

Here the coefficients A and B are real-valued functions that depend on model parameters and the time step Δ, and satisfy the initial conditions: $A(0,0;(k-1)\Delta,k\Delta) = B(0,0;(k-1)\Delta,k\Delta) = 0$, for any $k = 1,\cdots,nT$.[4] In order to ensure that $h_{k\Delta}$ represents the conditional variance of $y_{k\Delta}$ per time step of length Δ, A and B should also verify that $\left(\partial^2 A(\phi,0;(k-1)\Delta,k\Delta)/\partial\phi^2\right)_{\phi=0} = 0$ and $\left(\partial^2 B(\phi,0;(k-1)\Delta,k\Delta)/\partial\phi^2\right)_{\phi=0} = \Delta$.

In the affine GARCH option pricing literature, it is sometimes more convenient to work with quantities related to the log-asset price process $Y_{k\Delta}$, thus we rewrite Eq. (2.3) in the following equivalent way:

$$C_{(Y_{k\Delta},h_{(k+1)\Delta})}\left(\phi,\psi \mid \mathcal{F}_{(k-1)\Delta}\right) = A(\phi,\psi;(k-1)\Delta,k\Delta) + \phi Y_{(k-1)\Delta} + B(\phi,\psi;(k-1)\Delta,k\Delta)\, h_{k\Delta}. \tag{2.4}$$

Using (2.4), we can now characterize the multi-step conditional bivariate c.g.f. of $Y_{k\Delta}$ and $h_{(k+1)\Delta}$ in the proposition below.

Proposition 2.1 *For any $l < k$, with $k = 1,\ldots,nT-1$, the joint c.g.f. of $\left(Y_{k\Delta}, h_{(k+1)\Delta}\right)$ conditional on $\mathcal{F}_{l\Delta}$ is given by:*

$$C_{(Y_{k\Delta},h_{(k+1)\Delta})}\left(\phi,\psi \mid \mathcal{F}_{l\Delta}\right) = A(\phi,\psi;l\Delta,k\Delta) + \phi Y_{l\Delta} + B(\phi,\psi;l\Delta,k\Delta)\, h_{(l+1)\Delta}. \tag{2.5}$$

Here the coefficients $A(\phi,\psi;l\Delta,k\Delta)$ and $B(\phi,\psi;l\Delta,k\Delta)$ satisfy the following recursions:

$$A(\phi,\psi;l\Delta,k\Delta) = A(\phi,\psi;(l+1)\Delta,k\Delta) + A(\phi,B(\phi,\psi;(l+1)\Delta,k\Delta);l\Delta,(l+1)\Delta). \tag{2.6}$$

$$B(\phi,\psi;l\Delta,k\Delta) = B(\phi,B(\phi,\psi;(l+1)\Delta,k\Delta);l\Delta,(l+1)\Delta), \tag{2.7}$$

with the terminal conditions $A(\phi,\psi;k\Delta,k\Delta) = 0$ and $B(\phi,\psi;k\Delta,k\Delta) = \psi$, for any real-valued ϕ and ψ.

Proof See Section A.1 in the "Appendix". □

The next step is to characterize the risk-neutral affine representation of the asset returns and conditional variance. Since the market described by the general model from (2.1)–(2.2) is incomplete, we need to identify a pricing kernel candidate. We follow the standard approach proposed in the discrete-time affine framework literature (see e.g. Christoffersen et al. (2013a), Majewski et al. (2015), Khrapov and Renault (2016) among others) and use a variance dependent pricing kernel of the following form:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := \prod_{k=1}^{nT} \exp\left(\eta_1(\Delta) y_{k\Delta} + \eta_2(\Delta) h_{(k+1)\Delta} - C_{(y_{k\Delta},h_{(k+1)\Delta})}\left(\eta_1(\Delta),\eta_2(\Delta) \mid \mathcal{F}_{(k-1)\Delta}\right)\right). \tag{2.8}$$

Here $\eta_1(\Delta)$ and $\eta_2(\Delta)$ are the market prices of equity and variance risk, respectively, which, unlike in the non-affine GARCH framework of Badescu et al. (2017), are assumed to depend only on the time step Δ, but not on k. As further explained in Badescu et al. (2017), allowing the latter dependence may alter the affine structure of the model. The normalizing constant

[4] Note that the functions $A(0,0;(k-1)\Delta,k\Delta)$ and $B(0,0;(k-1)\Delta,k\Delta)$ defined in (2.3) do not depend on k, but we choose to keep this notation because later on we introduce an exponential affine structure for the multi-step bivariate distribution of returns and conditional variance and we shall use the same symbols for the corresponding coefficients.

$C_{(y_{k\Delta},h_{(k+1)\Delta})}\left(\eta_1(\Delta),\eta_2(\Delta)\mid\mathcal{F}_{(k-1)\Delta}\right)$ ensures that Q is a well-defined probability measure. Christoffersen et al. (2013a) showed that such a pricing kernel improves considerably the pricing performance of a Gaussian and Inverse Gaussian driven affine GARCH model relative to the standard conditional Esscher transform, which ignores the effect of a market price of variance risk.

We need to impose a condition on the two risk premium parameters $\eta_1(\Delta)$ and $\eta_2(\Delta)$, such that the no-arbitrage conditions are satisfied (i.e. Q is an equivalent martingale measure with respect to P). More specifically, we require that the discounted asset price process is a martingale under the risk-neutral measure Q, which is equivalent to having $\mathrm{E}^Q\left[\exp\left(y_{k\Delta}\right)\mid\mathcal{F}_{(k-1)\Delta}\right]=\exp\left(r\Delta\right)$ for any $k=1,\cdots,nT$, where r represents the one-period risk-free interest rate. An important advantage of the pricing kernel from (2.8) over other potential candidates is that it preserves the affine structure of the model after the change of measure. The next result characterizes the joint cumulant generating function of $y_{k\Delta}$ and $h_{(k+1)\Delta}$ under the risk-neutral measure Q.

Proposition 2.2 *The risk-neutral joint c.g.f. of* $\left(y_{k\Delta},h_{(k+1)\Delta}\right)$ *conditional on* $\mathcal{F}_{(k-1)\Delta}$ *is given by:*

$$C^Q_{(y_{k\Delta},h_{(k+1)\Delta})}\left(\phi,\psi\mid\mathcal{F}_{(k-1)\Delta}\right)=A^Q\left(\phi,\psi;(k-1)\Delta,k\Delta\right)+B^Q\left(\phi,\psi;(k-1)\Delta,k\Delta\right)h_{k\Delta}. \tag{2.9}$$

Here, the coefficients $A^Q\left(\phi,\psi;l\Delta,k\Delta\right)$ *and* $B^Q\left(\phi,\psi;l\Delta,k\Delta\right)$ *satisfy the following relationships:*

$$A^Q\left(\phi,\psi;(k-1)\Delta,k\Delta\right)=A\left(\phi+\eta_1(\Delta),\psi+\eta_2(\Delta);(k-1)\Delta,k\Delta\right)-A\left(\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right), \tag{2.10}$$

$$B^Q\left(\phi,\psi;(k-1)\Delta,k\Delta\right)=B\left(\phi+\eta_1(\Delta),\psi+\eta_2(\Delta);(k-1)\Delta,k\Delta\right)-B\left(\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right), \tag{2.11}$$

and the market prices of risk $\eta_1(\Delta)$ *and* $\eta_2(\Delta)$ *satisfy the no-arbitrage constraints below:*

$$A\left(1+\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right)=A\left(\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right)+r\Delta, \tag{2.12}$$

$$B\left(1+\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right)=B\left(\eta_1(\Delta),\eta_2(\Delta);(k-1)\Delta,k\Delta\right). \tag{2.13}$$

Proof See Section A.2 in the “Appendix”. □

Following Proposition 2.1, we can obtain an affine formulation for the conditional multi-step joint c.g.f. of $\left(Y_{k\Delta},h_{(k+1)\Delta}\right)$ under Q, with the coefficients satisfying recursions similar to those in (2.6)–(2.7). However, as we shall see in the later section, it is more convenient to work with quantities related to the risk-neutral conditional variance instead of $h_{(k+1)\Delta}$.

Finally, we notice that although we are able to characterize the risk-neutral joint c.g.f. in terms of the exponential affine coefficients under P, we cannot provide explicit risk-neutral asset return dynamics similar to those from (2.1)–(2.2), unless we specify both the P-innovation distribution and the volatility function f_3. These are provided for the special model (i.e. the HN-GARCH model) considered in the next subsection.

Our numerical analysis shall be based on a special case of the more general setting introduced in (2.1)–(2.2), which obeys the affine representation from (2.3). More specifically, we consider the Gaussian affine GARCH model of Heston and Nandi (2000a).

2.2 The affine Gaussian GARCH model (HN-GARCH model)

We assume that $y_{k\Delta}$ follows a discretized version of the Heston and Nandi (2000a) GARCH(1,1) model under the historical measure P:

$$y_{k\Delta} = (r + \lambda h_{k\Delta})\,\Delta + \sqrt{\Delta}\sqrt{h_{k\Delta}}\epsilon_{k\Delta}, \quad \epsilon_{k\Delta} \overset{P}{\sim} \mathbf{N}(0, 1), \tag{2.14}$$

$$h_{(k+1)\Delta} = \omega(\Delta) + \beta(\Delta)h_{k\Delta} + \alpha(\Delta)\left(\epsilon_{k\Delta} - \gamma(\Delta)\sqrt{h_{k\Delta}}\right)^2. \tag{2.15}$$

Note that the above dynamics represents a special case of the model (2.1)–(2.2) if we take $f_1 = r + \lambda h_{k\Delta}$, $f_2 = \omega(\Delta) + \beta(\Delta)h_{k\Delta}$ and $f_3 = \alpha(\Delta)\left(\epsilon_{k\Delta} - \gamma(\Delta)\sqrt{h_{k\Delta}}\right)^2$. Here the parameter λ quantifies the market price of equity risk.[5] The GARCH innovations $\epsilon_{k\Delta}$ are assumed to be i.i.d. Gaussian distributed random variables. We assume that the parameters governing the conditional variance dynamic from (2.15) satisfy the usual constraints, which ensure the stationarity of the model. For instance, we impose that $\omega(\Delta)$, $\alpha(\Delta)$ and $\beta(\Delta)$ are non-negative, and the persistence satisfies $d(\Delta) = \beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) < 1$. The parameter $\gamma(\Delta)$ quantifies the leverage effect, which is a positive value implying a negative correlation between the asset return and the volatility level, as usually observed in the equity markets. Later on, we shall use specific representations of the time-dependent GARCH parameters, which allow us to compute the weak diffusion limits.

Using the fact that the joint c.g.f. of $\epsilon_{k\Delta}$ and $\epsilon^2_{k\Delta}$ is given by:

$$C_{\left(\epsilon_{k\Delta}, \epsilon^2_{k\Delta}\right)}(\phi, \psi) = -\frac{1}{2}\log(1 - 2\psi) + \frac{\phi^2}{2(1 - 2\psi)},$$

we can show that the one-step conditional joint c.g.f. of $\left(y_{k\Delta}, h_{(k+1)\Delta}\right)$ has the affine structure from (2.3) with the coefficients given by:

$$A(\phi, \psi; (k-1)\Delta, k\Delta) = \phi r\Delta + \psi\omega(\Delta) - \frac{1}{2}\log(1 - 2\psi\alpha(\Delta)), \tag{2.16}$$

$$B(\phi, \psi; (k-1)\Delta, k\Delta) = \phi\lambda\Delta + \psi\left(\alpha(\Delta)\gamma^2(\Delta) + \beta(\Delta)\right) + \frac{\left(\phi\sqrt{\Delta} - 2\psi\alpha(\Delta)\gamma(\Delta)\right)^2}{2(1 - 2\psi\alpha(\Delta))}. \tag{2.17}$$

Using (2.16)–(2.17), we can take $\psi = 0$ in (2.3) and verify that the one-step conditional c.g.f. of $y_{k\Delta}$ corresponds to that of a Gaussian distribution:

$$C_{y_{k\Delta}}\left(\phi \mid \mathcal{F}_{(k-1)\Delta}\right) = \phi(r + \lambda h_{k\Delta})\,\Delta + \frac{\phi^2}{2}h_{k\Delta}\Delta. \tag{2.18}$$

[5] Note that various studies (e.g. Christoffersen et al. (2013a) consider an adjustment to the conditional mean return which is given by the c.g.f. of the GARCH innovations evaluated at $\sqrt{h_{k\Delta}}$. In the Gaussian case, this implies that λ is replaced by $\lambda - 1/2$. However, since this does not affect our theoretical results, we do not consider this adjustment.

From Proposition 2.1, the function $C_{(Y_{k\Delta}, h_{(k+1)\Delta})}(\phi, \psi \mid \mathcal{F}_{l\Delta})$ has the affine representation from (2.25) with the multi-step coefficients satisfying the recursions for any $l < k$:

$$\begin{aligned} A(\phi, \psi; l\Delta, k\Delta) &= A(\phi, \psi; (l+1)\Delta, k\Delta) + \phi r \Delta + \omega(\Delta) B(\phi, \psi; (l+1)\Delta, k\Delta) \\ &\quad - \frac{1}{2} \log\left(1 - 2\alpha(\Delta) B(\phi, \psi; (l+1)\Delta, k\Delta)\right). \end{aligned} \tag{2.19}$$

$$\begin{aligned} B(\phi, \psi; l\Delta, k\Delta) &= \phi\lambda\Delta + \left(\alpha(\Delta)\gamma^2(\Delta) + \beta(\Delta)\right) B(\phi, \psi; (l+1)\Delta, k\Delta) \\ &\quad + \frac{\left(\phi\sqrt{\Delta} - 2\alpha(\Delta)\gamma(\Delta) B(\phi, \psi; (l+1)\Delta, k\Delta)\right)^2}{2\left(1 - 2\alpha(\Delta) B(\phi, \psi; (l+1)\Delta, k\Delta)\right)}, \end{aligned} \tag{2.20}$$

together with the terminal conditions $A(\phi, \psi; k\Delta, k\Delta) = 0$ and $B(\phi, \psi; k\Delta, k\Delta) = \psi$, for any real-valued ϕ and ψ. The above recursions can be viewed as generalizations of some of the well-known results previously obtained for the HN-GARCH model (refer to (2.14)–(2.15)). Indeed, if we let $\Delta = 1$ and take $\phi = 0$ in (2.19) and (2.20), we recover the same recursions for the moment generating function of the log-price process Y_k as in Heston and Nandi (2000a).[6] Similarly, if we take $\psi = 0$ in both (2.19) and (2.20) with $\Delta = 1$, we recover the recursions for the moment generating functions of the variance process derived by Heston and Nandi (2000b).

In order to characterize the risk-neutral dynamics of the asset returns, we first need to identify the relationship between the market prices of risk, which satisfy the no-arbitrage constraints. Thus, if we replace the expressions for A and B from (2.16)–(2.17) into (2.13), we obtain:

$$\eta_1(\Delta) = -\lambda - \frac{1}{2} + 2\alpha(\Delta)\eta_2(\Delta)\left(\lambda + \frac{\gamma(\Delta)}{\sqrt{\Delta}}\right). \tag{2.21}$$

Note that (2.12) is automatically satisfied for any pair $(\eta_1(\Delta), \eta_2(\Delta))$. The risk-neutral asset returns dynamics are illustrated in the following corollary.

Corollary 2.1 *Suppose that the asset returns follow the affine GARCH dynamics from* (2.14)–(2.15) *under* P. *Then, the risk-neutral dynamics under the variance dependent pricing kernel from* (2.8) *are given by:*

$$y_{k\Delta} = \left(r - \frac{h^*_{k\Delta}}{2}\right)\Delta + \sqrt{\Delta}\sqrt{h^*_{k\Delta}}\epsilon^*_{k\Delta}, \quad \epsilon^*_{k\Delta} \overset{Q}{\sim} \mathbf{N}(0,1), \tag{2.22}$$

$$h^*_{(k+1)\Delta} = \omega^*(\Delta) + \beta(\Delta)h^*_{k\Delta} + \alpha^*(\Delta)\left(\epsilon^*_{k\Delta} - q^*(\Delta)\sqrt{h^*_{k\Delta}}\right)^2. \tag{2.23}$$

Here the leverage parameter is defined by $q^*(\Delta) = \lambda^*(\Delta) + \gamma^*(\Delta) + \frac{\sqrt{\Delta}}{2}$ *with the risk-neutral GARCH parameters governing the conditional variance equation defined below:*

$$\begin{aligned} \omega^*(\Delta) &= \frac{\omega(\Delta)}{1 - 2\alpha(\Delta)\eta_2(\Delta)}; \quad \alpha^*(\Delta) = \frac{\alpha(\Delta)}{(1 - 2\alpha(\Delta)\eta_2(\Delta))^2}; \\ \alpha^*(\Delta) &= \frac{\alpha(\Delta)}{(1 - 2\alpha(\Delta)\eta_2(\Delta))^2}; \\ \lambda^*(\Delta) &= \lambda\sqrt{\Delta}(1 - 2\alpha(\Delta)\eta_2(\Delta)); \qquad \gamma^*(\Delta) = \gamma(\Delta)(1 - 2\alpha(\Delta)\eta_2(\Delta)). \end{aligned}$$

[6] As pointed out by Christoffersen et al. (2013b), the formula for the moment generating function in Heston and Nandi (2000a) contains some typos. The correct relationship is illustrated in the former paper and is thus obtained as a special case of our result when $\Delta = 1$ and $\phi = 0$.

Proof See Section A.3 in the "Appendix". □

When $\Delta = 1$, the discretized risk-neutral GARCH model from (2.22)–(2.23) reduces to the model of Christoffersen et al. (2013a).[7] We notice that the asset return's risk-neutral conditional variance, $h^*_{k\Delta} = Var^Q[Y_{k\Delta}|\mathcal{F}_{(k-1)\Delta}]$, is related to the physical one by the following relationship:

$$h^*_{k\Delta} = \frac{h_{k\Delta}}{1 - 2\alpha(\Delta)\eta_2(\Delta)}. \tag{2.24}$$

Thus, a positive value for the market price of variance risk, $\eta_2(\Delta) > 0$, implies that $h^*_{k\Delta} \geq h_{k\Delta}$ as $\alpha(\Delta) \geq 0$. When $\eta_2(\Delta) = 0$, the above pricing kernel reduces to the standard conditional Esscher transform, and thus the above dynamics coincide with those in Heston and Nandi (2000a).

Since the GARCH dynamics under P and Q have the same form, we can use the results from Proposition 2.2 and equations (2.19)–(2.20) to characterize the multi-step risk-neutral c.g.f. of $Y_{k\Delta}$ and $h^*_{(k+1)\Delta}$:

$$C^Q_{\left(Y_{k\Delta}, h^*_{(k+1)\Delta}\right)} (\phi, \psi \mid \mathcal{F}_{l\Delta}) = A^* (\phi, \psi; l\Delta, k\Delta) + \phi Y_{l\Delta} + B^* (\phi, \psi; l\Delta, k\Delta)\, h^*_{(l+1)\Delta}. \tag{2.25}$$

Here the coefficients $A^* (\phi, \psi; l\Delta, k\Delta)$ and $B^* (\phi, \psi; l\Delta, k\Delta)$ satisfy the following recursions for all $l < k$:

$$\begin{aligned} A^* (\phi, \psi; l\Delta, k\Delta) &= A^* (\phi, \psi; (l+1)\Delta, k\Delta) + \phi r \Delta + \omega^*(\Delta) B^* (\phi, \psi; (l+1)\Delta, k\Delta) \\ &\quad - \frac{1}{2} \log \left(1 - 2\alpha^*(\Delta) B^* (\phi, \psi; (l+1)\Delta, k\Delta)\right), \\ B^* (\phi, \psi; l\Delta, k\Delta) &= -\frac{\phi\Delta}{2} + \left(\alpha^*(\Delta)(q^*(\Delta))^2 + \beta(\Delta)\right) B^* (\phi, \psi; (l+1)\Delta, k\Delta) \\ &\quad + \frac{\left(\phi\sqrt{\Delta} - 2\alpha^*(\Delta) q^*(\Delta) B^* (\phi, \psi; (l+1)\Delta, k\Delta)\right)^2}{2\left(1 - 2\alpha^*(\Delta) B^* (\phi, \psi; (l+1)\Delta, k\Delta)\right)}, \end{aligned}$$

with the terminal conditions $A^* (\phi, \psi; k\Delta, k\Delta) = 0$ and $B^* (\phi, \psi; k\Delta, k\Delta) = \psi$, for any real-valued ϕ and ψ.

3 Variance swap valuation in affine GARCH models

Variance swaps are forward contracts written on the Realized Variance (RV) of an underlying asset. In this section, we consider the pricing of discretely sampled variance swaps within the affine GARCH framework and we provide closed-form expressions for the fair strike prices under the Gaussian innovations specification.

We assume that the variance swap is sampled at the *same* frequency as the observation frequency of the underlying GARCH model. In this case, the RV over the interval $[0, \ldots, T]$, which is based on nT sampling points, is defined as the annualized sum of one-period squared log returns:

[7] Note that the pricing kernel used in Christoffersen et al. (2013a) is based on four parameters which are determined by imposing the standard no-arbitrage conditions. Our pricing kernel contains only two parameters and the derivation of the risk-neutral dynamics is done using a different approach.

$$RV(0,T,nT) = \frac{1}{T}\sum_{k=1}^{nT}\left(\log\frac{S_{k\Delta}}{S_{(k-1)\Delta}}\right)^2 = \frac{1}{T}\sum_{k=1}^{nT}\left(Y_{k\Delta}-Y_{(k-1)\Delta}\right)^2. \tag{3.1}$$

Since there are no cash exchanges at inception, the fair strike price of the swap at time $t=0$, denoted here by $K(0,T,N)$, is given by the risk-neutral expectation of the realized variance:

$$K(0,T,nT) = \mathrm{E}^Q\left[RV(0,T,nT)\right]. \tag{3.2}$$

In our case, the risk-neutral measure Q is constructed based on the variance-dependent kernel in (2.8). In general, depending on the risk-neutral asset returns dynamics, the expectation in (3.2) can be computed using different methods. For example, one can undertake a direct approach by evaluating the squared return process in (3.1) and calculating its risk-neutral expectation according to the corresponding dynamics. However, this procedure may be deemed as less general since the derivations depend heavily on the innovation distribution and on the risk-neutral conditional variance equation.

A more general approach is based on the methodology proposed by Hong (2004), which relies on computing the second moment by differentiating the (conditional) risk-neutral cumulant generating function of $Y_{k\Delta}-Y_{(k-1)\Delta}$, $C^Q_{Y_{k\Delta}-Y_{(k-1)\Delta}}(\phi \mid \mathcal{F}_{l\Delta})$, for any $k=1,\ldots,nT$ and $l<k$, while making use of the well-known relationship between moments and cumulants. This approach is rather general in our affine setting since the multi-step conditional risk-neutral bivariate cumulant generating function of $Y_{k\Delta}$ and $h_{(k+1)\Delta}$ (or $h^*_{(k+1)\Delta}$) is available in closed-form with coefficients satisfying explicit recursion relations. This can be seen in the lemma below.

Lemma 3.1 *Suppose that the risk-neutral joint c.g.f. of $Y_{k\Delta}$ and $h^*_{(k+1)\Delta}$, $C^Q_{\left(Y_{k\Delta},h^*_{(k+1)\Delta}\right)}(\phi,\psi \mid \mathcal{F}_{l\Delta})$, is affine in $h^*_{(l+1)\Delta}$, with the coefficients $A^*(\phi,\psi;l\Delta,k\Delta)$ and $B^*(\phi,\psi;l\Delta,k\Delta)$ satisfying the risk-neutral versions of the recursions in (2.6)–(2.7), for any $l<k$. Then, the risk-neutral c.g.f. of $Y_{k\Delta}-Y_{(k-1)\Delta}$ conditional on $\mathcal{F}_{l\Delta}$ has the following affine representation:*

$$\begin{aligned} C^Q_{Y_{k\Delta}-Y_{(k-1)\Delta}}(\phi \mid \mathcal{F}_{l\Delta}) = {} & A^*(\phi,0;(k-1)\Delta,k\Delta) \\ & + A^*\left(0, B^*(\phi,0;(k-1)\Delta,k\Delta); l\Delta,(k-1)\Delta\right) \\ & + B^*\left(0, B^*(\phi,0;(k-1)\Delta,k\Delta); l\Delta,(k-1)\Delta\right) h^*_{(l+1)\Delta}. \end{aligned}$$

Proof See Section A.4 in the "Appendix". □

Following the above result, we can now compute the nth order multi-step conditional cumulants of the log-return process $Y_{k\Delta}-Y_{(k-1)\Delta}$, by evaluating the nth order derivatives of $C^Q_{Y_{k\Delta}-Y_{(k-1)\Delta}}(\phi \mid \mathcal{F}_{l\Delta})$ at $\phi=0$. Thus, the valuation of the variance swaps reduces to the computation of the first and second order partial derivatives of the affine coefficients A^* and B^* evaluated at different time points. The key ingredient in such a derivation is provided by the risk-neutral versions of the recursions in (2.6)–(2.7).

From Lemma 3.1, we can write the unconditional (i.e. $l=0$) cumulant generating function of the one-period log-return process $y_{k\Delta}$ as:

$$\begin{aligned} C^Q_{y_{k\Delta}}(\phi) = {} & A^*(\phi,0;(k-1)\Delta,k\Delta) + A^*\left(0, B^*(\phi,0;(k-1)\Delta,k\Delta); 0,(k-1)\Delta\right) \\ & + B^*\left(0, B^*(\phi,0;(k-1)\Delta,k\Delta); 0,(k-1)\Delta\right) h^*_{\Delta}. \end{aligned} \tag{3.3}$$

Since the nth cumulants of $y_{k\Delta}$ are defined by $\kappa^{(n)}_{y_{k\Delta}} := \partial^n C^Q_{y_{k\Delta}}(\phi)/\partial\phi^n \mid_{\phi=0}$, we need to compute the first and second order derivatives of the coefficients from (3.3). For any

$i, j = 0, 1, 2$ and $k = 1, \ldots, nT$, we introduce the following useful notations for these derivatives corresponding to the one-step coefficients:

$$A^*_{ij} = \frac{\partial^{i+j} A^*(\phi, \psi; (k-1)\Delta, k\Delta)}{\partial \phi^i \partial \psi^j} |_{\phi=0,\psi=0},$$
$$B^*_{ij} = \frac{\partial^{i+j} B^*(\phi, \psi; (k-1)\Delta, k\Delta)}{\partial \phi^i \partial \psi^j} |_{\phi=0,\psi=0} . \tag{3.4}$$

Note that both A^*_{ij} and B^*_{ij} depend on the time step Δ, but to notationally simplify the final expressions we omit expressing this dependence. In the Appendix we show how the first and second order derivatives of the multi-step coefficients from (3.3) can be expressed in terms of (3.4), so the variance swap price will also depend on these terms. The main result of this subsection is contained in the following proposition.

Proposition 3.1 *Suppose that the risk-neutral joint c.g.f. of $Y_{k\Delta}$ and $h^*_{(k+1)\Delta}$, $C^Q_{\left(Y_{k\Delta}, h^*_{(k+1)\Delta}\right)}(\phi, \psi \mid \mathcal{F}_{l\Delta})$, is affine in $h^*_{(l+1)\Delta}$, with the coefficients $A^*(\phi, \psi; l\Delta, k\Delta)$ and $B^*(\phi, \psi; l\Delta, k\Delta)$ satisfying the recursions (risk-neutral versions) from (2.6)–(2.7) for any $l < k$. Then, the fair strike price of the discretely sampled variance swap defined in (3.2) is given by:*

$$K(0, T, nT) = nF_1 + \frac{F_2}{T}\frac{1 - B^{*nT}_{01}}{1 - B^*_{01}} + \frac{F_3}{T}\frac{1 - B^{*2nT}_{01}}{1 - B^{*2}_{01}}$$
$$+ \left(\frac{F_4}{T}\frac{1 - B^{*nT}_{01}}{1 - B^*_{01}} + \frac{F_5}{T}\frac{1 - B^{*2nT}_{01}}{1 - B^{*2}_{01}}\right) h^*_\Delta + \frac{F_6}{T}\frac{1 - B^{*2nT}_{01}}{1 - B^{*2}_{01}} h^{*2}_\Delta. \tag{3.5}$$

Here, the factors $F_1 - F_6$ are given below:

$$F_1 = A^{*2}_{10} + A^*_{20} + \frac{2A^*_{01}A^*_{10}B^*_{10} + A^*_{01}B^*_{20}}{1 - B^*_{01}} + \frac{A^{*2}_{01}B^{*2}_{10}}{\left(1 - B^*_{01}\right)^2} + \frac{A^*_{02}B^{*2}_{10}}{1 - B^{*2}_{01}}$$
$$+ \frac{A^*_{01}B^*_{02}B^{*2}_{10}}{\left(1 - B^*_{01}\right)\left(1 - B^{*2}_{01}\right)}, \tag{3.6}$$

$$F_2 = -\frac{2A^*_{01}A^*_{10}B^*_{10} + A^*_{01}B^*_{20}}{1 - B^*_{01}} - \frac{2A^{*2}_{01}B^{*2}_{10}}{\left(1 - B^*_{01}\right)^2} - \frac{A^*_{01}B^*_{02}B^{*2}_{10}}{B^*_{01}\left(1 - B^*_{01}\right)^2}, \tag{3.7}$$

$$F_3 = \frac{A^{*2}_{01}B^{*2}_{10}}{\left(1 - B^*_{01}\right)^2} - \frac{A^*_{02}B^{*2}_{10}}{1 - B^{*2}_{01}} + \frac{A^*_{01}B^*_{02}B^{*2}_{10}}{B^*_{01}\left(1 - B^*_{01}\right)\left(1 - B^{*2}_{01}\right)}, \tag{3.8}$$

$$F_4 = B^*_{20} + 2A^*_{10}B^*_{10} + \frac{B^{*2}_{10}\left(B^*_{02} + 2A^*_{01}B^*_{01}\right)}{B^*_{01}\left(1 - B^*_{01}\right)}, \tag{3.9}$$

$$F_5 = -\frac{B^{*2}_{10}\left(B^*_{02} + 2A^*_{01}B^*_{01}\right)}{B^*_{01}\left(1 - B^*_{01}\right)}, \tag{3.10}$$

$$F_6 = B^{*2}_{10}. \tag{3.11}$$

Proof See Section A.5 in the "Appendix". □

We notice that the variance swap fair strike is a quadratic function of h^*_Δ, whose coefficients depend both on the GARCH parameters and the time step Δ. According to the above result, the variance swap valuation problem thus reduces to identifying and computing the partial derivatives at $\phi = \psi = 0$ of the one step coefficients $A^*(\phi, \psi; (k-1)\Delta, k\Delta)$

and $B^*(\phi, \psi; (k-1)\Delta, k\Delta)$ from the affine representation of the bivariate c.g.f. of $y_{k\Delta}$ and $h^*_{(k+1)\Delta}$ under Q. This is presented in the following corollary for the HN-GARCH model:

Corollary 3.1 *If the asset price follows the risk-neutral dynamics from (2.22)–(2.23), then the fair strike price of the discretely sampled variance swap defined in (3.2) is given in (3.5) with the one-step coefficients given by:*

$$A^*_{10} = r\Delta; \quad A^*_{20} = 0; \quad B^*_{10} = -\frac{\Delta}{2}; \quad B^*_{20} = \Delta,$$
$$A^*_{01} = \omega^*(\Delta) + \alpha^*(\Delta); \quad A^*_{02} = 2\left(\alpha^*(\Delta)\right)^2; \quad B^*_{01} = d^*(\Delta); \quad B^*_{02} = 4\left(\alpha^*(\Delta)\right)^2\left(q^*(\Delta)\right)^2, \tag{3.12}$$

where $d^*(\Delta) = \beta(\Delta) + \alpha^*(\Delta)(q^*(\Delta))^2$ *represents the risk-neutral persistence of the GARCH model.*

Proof See Section A.6 in the "Appendix". □

The variance swap price for the HN-GARCH model is thus obtained as a special case of the result in Proposition 3.1 by replacing the one-step coefficients derived in (3.12) into the general formula (3.5). For verification purposes, we shall also present in the Appendix A.6 an alternative derivation of the strike price for the HN-GARCH model, which consists of computing (3.2) through a direct evaluation of the risk-neutral expectation.

4 Variance swaps for GARCH diffusion limits and convergence

The aim of this section is to compute the prices of continuously sampled variance swaps based on the weak diffusion limits of the HN-GARCH model and to investigate the convergence of the discretely sampled GARCH swap rate to its continuous counterpart.

4.1 The HN-GARCH diffusion limit under the variance dependent pricing kernel

We follow the standard approach in Nelson (1990) to derive the weak limit of the affine GARCH model under the risk-neutral measure introduced in (2.8). First, we compute the GARCH weak diffusion limit under the physical measure P as:[8]

$$dY_t = (r + \lambda h_t)dt + \sqrt{h_t}dW_{1t}, \tag{4.1}$$
$$dh_t = \kappa(\theta - h_t)dt + \sigma\sqrt{h_t}dW_{2t}. \tag{4.2}$$

Here W_{1t} and W_{2t} are two correlated standard Brownian motions under P with $\mathrm{E}[dW_{1t}dW_{2t}] = \rho dt$ and $\rho = -1$. The diffusion limit parameters depend on the GARCH counterparts via the following relationships:

$$\theta = \frac{\omega + \alpha}{\kappa}, \quad \sigma = 2\alpha\gamma, \quad \kappa = 1 - \beta - \alpha\gamma^2. \tag{4.3}$$

The resulting affine diffusion limit is of Heston type with perfectly (negatively) correlated Brownian motions. A similar limit has been obtained in Heston and Nandi (2000a) for

[8] In the Appendix, we shall illustrate the proof for the weak limit of the Gaussian GARCH model only under the risk-neutral measure. The same approach is used under P, and since the proof is even simpler, we omit it from our presentation.

Gaussian innovations, although there are significant differences in the parametric constraints, which we explain in detail below.

The above diffusion coefficients are computed based on the following assumptions we impose on the GARCH parameters.

$$\omega(\Delta) = \omega\Delta; \quad \alpha(\Delta) = \alpha\Delta; \quad \gamma(\Delta) = \frac{\gamma}{\sqrt{\Delta}}; \quad \beta(\Delta) = 1 - \kappa\Delta - \alpha(\Delta)\gamma^2(\Delta). \quad (4.4)$$

When $\Delta = 1$ and if we define $\beta := \beta(1)$, an inspection of (2.14)–(2.15) shows that these parameters correspond to those for the dynamics of the Heston and Nandi (2000a) model sampled at a daily frequency. This observation has been overlooked in that study since their continuous time limit is computed based on choosing $\beta(\Delta) = 0$ (combined with other specifications for the rest of the parameters) for all frequencies, and this is not consistent with a GARCH structure when $\Delta = 1$. Our approach is therefore different, since we consider another asymptotic behavior that preserves the GARCH structure at a daily frequency. Although we use different parametric constraints, our weak diffusion limit is very similar to that in Heston and Nandi (2000a). The implied diffusion parameters satisfy the standard constraints in the Heston (1993) model. For instance, the GARCH stationarity constraint implies that $\kappa > 0$. Moreover, $\sigma > 0$ when the leverage effect γ is positive (this typically holds for most financial time-series) which allows for a negative correlation between the asset returns and variance. As shown in Badescu et al. (2017), the sign of the leverage effect is directly related to the presence of asset bubbles.

Following the same parametric constraints as under P, we derive the risk-neutral GARCH weak diffusion limit below, and summarize it in the following result.

Proposition 4.1 *Suppose that the market price of variance risk does not depend on the sampling frequency, that is, $\eta_2(\Delta) = \eta_2$. Then, under the parametric constraints in (4.4) for the HN-GARCH model, the weak diffusion limit of the risk neutral GARCH processes in (2.22)–(2.23) is given by the following square-root model:*

$$dY_t = \left(r - \frac{h_t}{2}\right)dt + \sqrt{h_t}dW^*_{1t}, \quad (4.5)$$

$$dh_t = \kappa^*(\theta^* - h_t)dt + \sigma\sqrt{h_t}dW^*_{2t}. \quad (4.6)$$

Here W^*_{1t} and W^*_{2t} are two correlated standard Brownian motions under Q with $\mathrm{E}^*[dW^*_{1t}dW^*_{2t}] = \rho dt$ and $\rho = -1$. The variance equation parameters depend on the GARCH counterparts in the following way:

$$\theta^* = \frac{\omega + \alpha}{\kappa^*}, \quad \sigma = 2\alpha\gamma, \quad \kappa^* = \kappa - \sigma\left(\lambda + \frac{1}{2}\right).$$

Proof See Section A.7 in the “Appendix”. □

Letting Δ converge to zero in (2.24), we notice that unlike in the discrete time case, the continuous-time risk-neutral variance is the same as the physical one. In fact, this can also be viewed as a consequence of Girsanov’s Theorem applied to the historical dynamics in (4.1)–(4.2) for a market price of equity risk $(\lambda + 1/2)\sqrt{h_t}$. Moreover, the variance dependent kernel parameter $\eta_2(\Delta) = \eta_2$ does not have any effect on the above continuous-time risk-neutral dynamics in (4.5)–(4.6), therefore the same limit is also recovered from the GARCH risk-neutralized dynamics obtained based on the conditional Esscher transform (i.e. $\eta_2(\Delta) = 0$). This limit corresponds to that obtained in Heston and Nandi (2000a) based on their parametric conditions.

Note that the only other scenario which leads to a finite risk-neutral variance, corresponds to taking a market price of variance risk of the form $\eta_2(\Delta) = \eta_2/\Delta$.[9] In this case, equation (2.24) implies that $h^*_{k\Delta} = h_{k\Delta}/(1-\alpha\eta_2)$, so that the corresponding continuous-time limit of the risk-neutral variance becomes $h_t/(1-\alpha\eta_2)$. However, the latter relationship would come into a contradiction with Girsanov's Principle since the asset returns variance remains unchanged after the change of measure, and hence the P and Q dynamics of the GARCH diffusion limits are inconsistent.

4.2 Variance swaps for continuous time limits and convergence results

In this section, we compute the discretely and continuously sampled variance swap fair strike prices when the underlying follows the above HN-GARCH weak diffusion limit dynamics.

According to a well-known asymptotic result (see e.g. Jacod and Protter (1998) and Andersen et al. (2003)), when the time between observations is small, the sample realized variance converges in probability to the annualized Quadratic Variation (QV) of the log-price, which, in our stochastic volatility framework, coincides with the integrated variance. In other words, for any partition $0 = t_0 < \cdots < t_N = T$ of the interval $[0, T]$ with $\max_{i=1,\ldots,N} |t_i - t_{i-1}| \to 0$, we have:

$$RV(0,T,N) := \frac{AF}{N} \sum_{k=1}^{N} \left(Y_{t_k} - Y_{t_{k-1}}\right)^2 \longrightarrow QV(0,T) := \frac{1}{T}\int_0^T h_t dt. \tag{4.7}$$

Here AF represents the annualization factor (e.g., if the sampling is performed daily (for every *trading day*), then AF = 252). In general, practitioners use the continuous sampling method in order to approximate the variance swap price when the underlying model is a diffusion process and this approximation is reasonably accurate if the realized variance is defined at a daily frequency (see e.g. Broadie and Jain (2008) and Jarrow et. al. (2013), among others). Our aim here is to show the convergence of the swap rate constructed based on the RV definition according to the left hand side of (4.7), where $(Y_{k\Delta}, h_{(k+1)\Delta})$ follows the affine HN-GARCH model, to the variance swap price based on the QV definition according to right hand side of (4.7), and when (Y_t, h_t) follows the diffusion limit of the HN-GARCH process. This is contained in the following proposition:

Proposition 4.2 *The following statements hold:*

(i) If the asset prices follow the stochastic volatility risk-neutral dynamics in (4.5)–(4.6), then the variance swap strike constructed based on the QV is given by:

$$\bar{K}(0,T) := \mathrm{E}^Q\left[QV(0,T)\right] = \theta^* + \left(h_0 - \theta^*\right)\frac{1-e^{-\kappa^* T}}{\kappa^* T}. \tag{4.8}$$

(ii) Suppose that the parametric constraints in (4.4) are satisfied. Then, as the time step Δ approaches zero (or equivalently as n approaches infinity), the fair strike price $K(0,T,nT)$, computed according to the specifications in Corollary 3.1, converges to the corresponding diffusion-based price $\bar{K}(0,T)$ stated above.

Proof See Section A.8 in the "Appendix". □

[9] Indeed, if we take $\eta_2(\Delta) = \eta_2/\Delta^\delta$, with $0 < \delta < 1$, we recover the same limit as in Proposition 4.1. When $\delta > 1$, the continuous-time limit of the risk-neutral variance is not well defined.

The first result in Proposition 4.2 is standard in the variance swap literature. Indeed, under the model assumptions in part (i), which implies that the asset risk-neutral dynamics follows a Heston type model with perfectly correlated Brownian motions, the fair strike price for a variance swap has been already derived in the literature (see e.g. Broadie and Jain (2008) and Swishchuk (2013) among others), so our result can be viewed as a special case. Part (ii) establishes the main convergence result of this section for discretely sampled variance swaps based on the HN-GARCH model, $K(0, T, nT)$ from (3.5) and coefficients satisfying (3.12), to the continuously sampled swaps constructed based on the continuous time limit of the HN-GARCH model, $\bar{K}(0, T)$ computed in (4.8). Numerical examples regarding this convergence result are presented in Sect. 6.

5 Numerical examples

We now present two numerical exercises to support the theoretical results presented in the previous sections. First, using daily historical returns on the S&P 500 index, we construct the term structure of the variance swap rates using a rolling window estimation methodology. Second, using a randomly chosen parameters set from those obtained at the first step, we test the numerical convergence of the fair strike prices from GARCH to diffusions.

5.1 Term structure of variance swaps

The data set used for the estimation procedure consists of daily closing prices for the S&P 500 index from January 16, 1986 to December 15, 2011. The term structure of the variance swap rates is constructed according to the following rolling window procedure:

1. Estimate the Gaussian GARCH model in (2.14)–(2.15) at daily frequency (i.e. $\Delta = 1$) using Maximum Likelihood Estimation (MLE) based on the first 2520 observations from the above sample, namely from January 16, 1986 to January 4, 1996:

$$y_k = r + \lambda h_k + \sqrt{h_k}\epsilon_k, \quad \epsilon_k \overset{P}{\sim} \mathbf{N}(0, 1), \tag{5.1}$$

$$h_{k+1} = \omega + \beta h_k + \alpha \left(\epsilon_k - \gamma\sqrt{h_k}\right)^2. \tag{5.2}$$

2. Based on the parameters estimated at Step 1, compute the variance swap prices using the formulas in Proposition 3.1.
3. Shift by three months the estimation window, that is, drop the first three months of sample points from the current sample and add the same amount of daily returns starting from the remaining data set. Repeat the same estimation and sampling procedure until the end of the data set is reached.

Thus, we end up with 64 estimation exercises that yield 64 corresponding parameter sets, the last fitted period being December 13, 2001 - December 15, 2011. The evolution of the estimated GARCH parameters is illustrated in Fig. 1. The estimated GARCH parameters are in the same range as those typically reported in various study for the HN-GARCH models (see e.g. Christoffersen et al. (2013a) among others). For example, the time series are persistent as β ranges from 0.84 to 0.96, while the market risk premium parameter λ takes smaller values when the time series contain financial crisis periods.

For each parameter set, we compute the corresponding variance swap prices for maturities ranging from 1 to 30 months. Since the market price of variance risk parameter η_2 from the

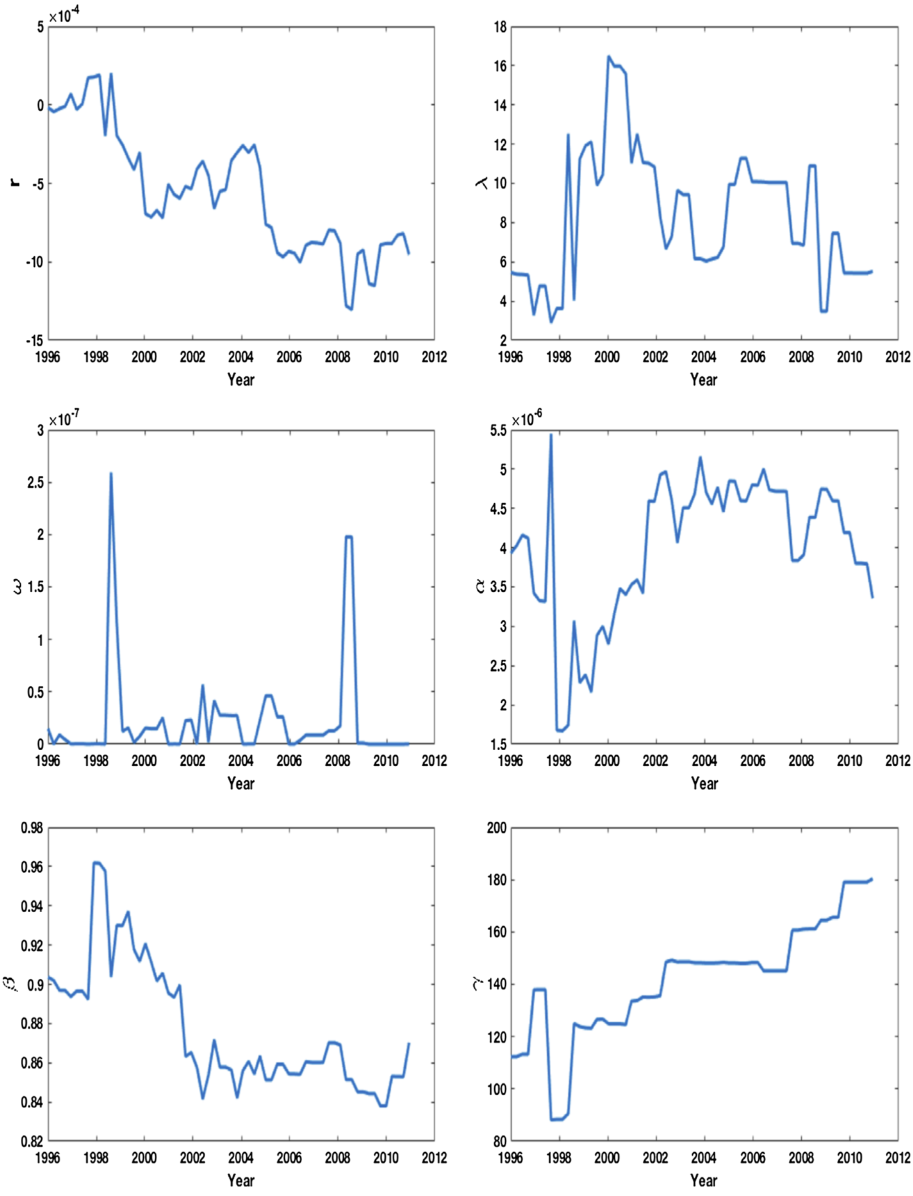


Fig. 1 HN-GARCH parameters estimated using a rolling window of 2520 daily closing prices for the S&P 500 index recorded over the period January 16, 1986 to December 15, 2011. The first exercise uses daily returns from January 16, 1986 to January 4, 1996, and the rolling window is constructed by dropping the first three months of sample points from the current sample and adding the same amount of daily returns starting from the remaining data set for a total of 64 estimations

pricing kernel (2.8) cannot be estimated from historical returns, and no other data sources have been used for this exercise, we investigate the behavior of the variance swap term structure with respect to this parameter. More specifically, we plot the fair strikes corresponding to market prices of variance risks computed for three levels of risk-neutral to physical conditional variance ratios, namely $h_k^*/h_k = 1, 1.25, 1.5$. Note that the first scenario $h_k^*/h_k = 1$ corresponds to the case of the conditional Esscher measure change since $\eta_2 = 0$ (see e.g. equation (2.24) for $\Delta = 1$). Results are illustrated in Fig. 2.

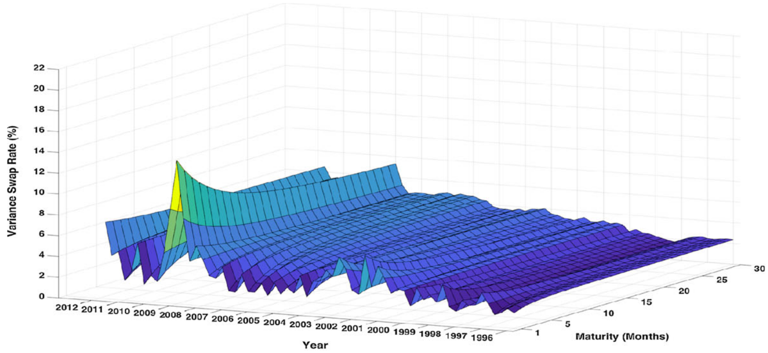
We notice that variance swaps exhibit some standard characteristics observed in real data. For example, in all three panels of Fig. 2, the rates are volatile, mean-reverting, with volatility clustering and spikes around the time of the financial crisis in 2008. Furthermore, as in Aït-Sahalia et al. (2015), we notice that the slope of the VS term structure, which is measured as the difference between the largest and the smallest term rates, are negatively correlated to the current volatility level. In particular, the VS exhibit a pronounced downward slope during highly volatile periods such as those in 2001–2002 and 2008. The amplitude of the term structure depends strongly on the variance risk premium parameter, η_2, in the pricing kernel. Panel (a) suggests that the smallest values for the fair strike prices correspond to the conditional Esscher transform, which corresponds to $\eta_2 = 0$. Panels (b) and (c) indicate that the VS term structure exhibit larger spikes for greater positive values of η_2, which also corresponds to having a U-shaped variance dependent pricing kernel. We emphasize that, in general, the variance risk premium parameter η_2 should be estimated from over the counter observed quotes of variance swap prices.

5.2 Convergence of variance swap prices

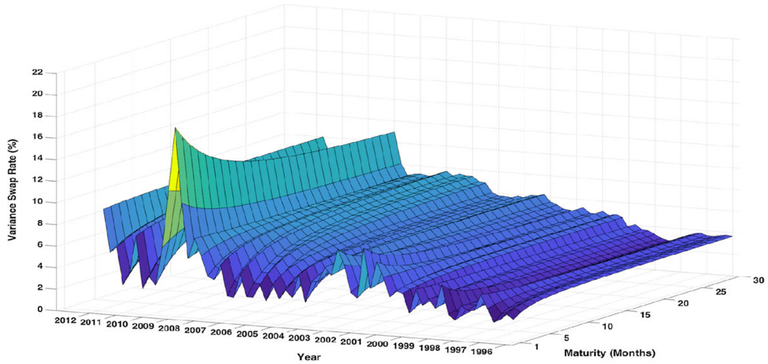
The numerical convergence of the GARCH-based variance swap prices constructed using the RV definition to the continuous time counterparts is illustrated using the following parametric specifications, which are taken from the last estimation exercise in the previous subsection (i.e. using daily historical returns from December 13, 2001–December 15, 2011). Therefore, we assume that the "daily" values of the GARCH parameters are: $r = -9.5395 \times 10^{-4}, \omega = 1.7244 \times 10^{-10}, \alpha = 3.3559 \times 10^{-6}, \beta = 0.8703, \gamma = 180.3003$ and $\lambda = 5.5083$. The values for the continuous-time diffusion limit parameters under P follow from (4.3) and the time-dependent GARCH parameters are related to the above daily values via (4.4). As in the previous exercise, we consider three values for the variance dependent pricing kernel parameter, which correspond to the same three levels of daily risk-neutral to physical conditional variance ratio, $\eta_2 = 0, \eta_2 = 2.9798 \times 10^4$ and $\eta_2 = 4.9663 \times 10^4$, respectively. The discrete-time variance swaps are computed for 11 values of intra-daily points, so that $n = 1, 2, 2^2, \ldots, 2^{10}$ and maturities ranging from 1 to 37 months. In the last line, we report the fair strike prices for the continuous variance swaps based on the GARCH diffusion limit. The results are reported in Tables 1, 2 and 3.

We notice that the discrete VS rates converge reasonably fast to the continuously sampled ones, with the largest differences between the two prices being recorded for the quantities computed at GARCH daily frequency, $\Delta = 1$. This difference is larger when η_2 increases, and this is a direct consequence of the fact that the continuous-time variance swap does not depend on the pricing kernel parameter. Moreover, the GARCH based VS rates depend on the starting value of the return process y_0 (or on the innovation shock ϵ_0), while the continuous counterpart does not have an explicit dependence on it. As Δ decreases, the influence of both η_2 and y_0 disappear, so the convergence is numerically established.

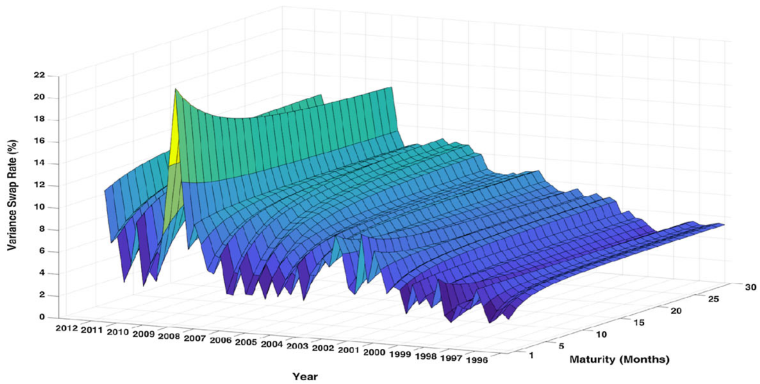
Another interesting aspect is that the convergence is realized from both below and above. For example, in Table 1, for the first three maturities, the convergence is from below, and for the remaining columns the convergence is from above. This justifies that the discretely



(a) Conditional Esscher transform with $h_k^* = h_k$



(b) Variance dependent pricing kernel with $h_k^* = 1.25h_k$



(c) Variance dependent pricing kernel with $h_k^* = 1.5h_k$

Fig. 2 Term structure of the variance swaps implied by daily historical returns on S&P 500 from January 16, 1986 to December 15, 2011. The fair strike prices are computed based on parameters estimated by MLE using a rolling window of 2520 observations. Each panel corresponds to the following pricing kernels: panel **a** corresponds to the Esscher transform obtained for $\eta_2 = 0$ and panels **b** and **c** correspond to the variance dependent pricing kernel obtained for values of η_2 which make $h_k^* = 1.25h_k$ and $h_k^* = 1.5h_k$, respectively. The variance swap prices are annualized and illustrated in variance percentage units. Maturity figures are in months

Table 1 Convergence of affine Gaussian GARCH based VS prices computed using the general formula in Proposition 3.1 and the specifications in Corollary 3.12 via the conditional Esscher transform ($\eta_2 = 0$)

Variance swap rates for GARCH models and diffusion limit (Esscher case; $h^*/h = 1$)

Intraday periods	Maturity in months									
	1	5	9	13	17	21	25	29	33	37
1	2.929	2.778	2.700	2.656	2.630	2.612	2.600	2.592	2.585	2.580
2	2.955	2.788	2.701	2.652	2.623	2.603	2.590	2.580	2.573	2.567
4	2.970	2.793	2.702	2.651	2.620	2.599	2.585	2.575	2.567	2.561
8	2.979	2.797	2.703	2.650	2.618	2.597	2.583	2.572	2.564	2.558
16	2.984	2.799	2.704	2.650	2.618	2.597	2.582	2.571	2.563	2.557
32	2.987	2.801	2.704	2.651	2.618	2.596	2.582	2.571	2.563	2.556
64	2.989	2.802	2.705	2.651	2.618	2.596	2.581	2.571	2.562	2.556
128	2.990	2.802	2.705	2.651	2.618	2.596	2.581	2.571	2.562	2.556
256	2.991	2.803	2.706	2.651	2.618	2.596	2.581	2.571	2.562	2.556
512	2.991	2.803	2.706	2.651	2.618	2.597	2.582	2.571	2.562	2.556
1024	2.992	2.803	2.706	2.651	2.618	2.597	2.582	2.571	2.562	2.556
Diffusion limit prices	2.992	2.804	2.706	2.652	2.618	2.597	2.582	2.571	2.562	2.556

These prices are calculated for different values of $n = 1, 2, \ldots, 2^{10}$ and maturities ranging from 1 to 37 months. The continuously sampled VS prices based on the GARCH diffusion limit in Proposition 4.2 are stated in the last line of the table. Parameters and starting values used in the computations are given in Sect. 5.2

Table 2 Convergence of affine Gaussian GARCH based VS prices computed using the general formula in Proposition 3.1 and the specifications in Corollary 3.12 via the conditional Esscher transform ($\eta_2 = 0$).

Variance swap rates for GARCH models and diffusion limit ($h^*/h = 1.25$)

Intraday periods	Maturity in months									
	1	5	9	13	17	21	25	29	33	37
1	3.761	3.854	3.902	3.930	3.946	3.957	3.965	3.970	3.975	3.978
2	3.324	3.248	3.208	3.186	3.172	3.164	3.157	3.153	3.150	3.147
4	3.145	3.008	2.937	2.897	2.873	2.857	2.846	2.838	2.832	2.827
8	3.064	2.901	2.816	2.769	2.740	2.721	2.708	2.699	2.692	2.686
16	3.026	2.850	2.759	2.709	2.678	2.658	2.644	2.633	2.626	2.619
32	3.008	2.826	2.732	2.679	2.647	2.627	2.612	2.601	2.593	2.587
64	2.999	2.814	2.719	2.665	2.633	2.611	2.597	2.586	2.578	2.571
128	2.995	2.809	2.712	2.658	2.625	2.604	2.589	2.578	2.570	2.563
256	2.993	2.806	2.709	2.655	2.622	2.600	2.585	2.574	2.566	2.560
512	2.992	2.805	2.707	2.653	2.620	2.598	2.583	2.572	2.564	2.558
1024	2.992	2.804	2.707	2.652	2.619	2.597	2.582	2.572	2.563	2.557
Diffusion limit prices	2.992	2.804	2.706	2.652	2.618	2.597	2.582	2.571	2.562	2.556

These prices are calculated for different values of $n = 1, 2, \ldots, 2^{10}$ and maturities ranging from 1 to 37 months. The continuously sampled VS prices based on the GARCH diffusion limit in Proposition 4.2 are stated in the last line of the table. Parameters and starting values used in the computations are given in Sect. 5.2

Table 3 Convergence of affine Gaussian GARCH based VS prices computed using the general formula in Proposition 3.1 and the specifications in Corollary 3.12 via the conditional Esscher transform ($\eta_2 = 0$).

Variance swap rates for GARCH models and diffusion limit ($h^*/h = 1.5$)

Intraday periods	Maturity in months									
	1	5	9	13	17	21	25	29	33	37
1	4.634	5.087	5.325	5.460	5.542	5.596	5.633	5.661	5.681	5.698
2	3.626	3.638	3.645	3.649	3.651	3.652	3.654	3.654	3.655	3.655
4	3.273	3.168	3.114	3.083	3.065	3.053	3.044	3.038	3.033	3.030
8	3.123	2.974	2.896	2.853	2.827	2.809	2.798	2.789	2.782	2.777
16	3.054	2.885	2.798	2.749	2.719	2.699	2.686	2.676	2.668	2.663
32	3.022	2.843	2.751	2.699	2.667	2.647	2.633	2.622	2.614	2.608
64	3.006	2.823	2.728	2.675	2.642	2.621	2.607	2.596	2.588	2.582
128	2.998	2.813	2.717	2.663	2.630	2.609	2.594	2.583	2.575	2.569
256	2.995	2.808	2.711	2.657	2.624	2.603	2.588	2.577	2.569	2.562
512	2.993	2.806	2.709	2.654	2.621	2.600	2.585	2.574	2.565	2.559
1024	2.993	2.805	2.707	2.653	2.620	2.598	2.583	2.572	2.564	2.557
Diffusion limit prices	2.992	2.804	2.706	2.652	2.618	2.597	2.582	2.571	2.562	2.556

These prices are calculated for different values of $n = 1, 2, \ldots, 2^{10}$ and maturities ranging from 1 to 37 months. The continuously sampled VS prices based on the GARCH diffusion limit in Proposition 4.2 are stated in the last line of the table. Parameters and starting values used in the computations are given in Sect. 5.2

sampled variance swap strikes may not always be higher than the continuously sampled variance swap strikes. This is consistent with the findings of Bernard and Cui (2014), where they obtain similar numerical evidence for the continuous-time stochastic volatility models. In Griessler and Keller-Ressel (2014), the following "Convex Order Conjecture" is proposed on page 3 of their paper:

$$\mathrm{E}\left[f(RV)\right] \geq \mathrm{E}\left[f(QV)\right], \tag{5.3}$$

for all convex functions f, i.e. realized variance dominates quadratic variation in convex order. Note that $f(x) = x$ is also in general a convex function, thus in order to propose a counterexample to this convex order conjecture, it is sufficient to seek counterexamples by comparing the fair strikes of discretely sampled and continuously sampled variance swaps. Bernard and Cui (2014) proposed numerical counterexamples showing that the discrete fair strikes can sometimes be lower than the continuous fair strikes and the convergence is from below. See also the additional discussions on page 17 of Griessler and Keller-Ressel (2014). Note that previous numerical counterexamples in the literature are all based on continuous time stochastic processes. Thus the numerical observations here for the HN-GARCH model provides a new class of numerical counterexamples based on discrete-time stochastic processes.

6 Conclusion

In this paper, we develop a general valuation framework for discretely sampled variance swaps in a general class of affine GARCH models. Using a variance dependent pricing kernel, which allows for an affine representation for the multi-step joint conditional cumulant generating function of the log-price and variance processes with coefficients satisfying general recursion relationships, we derive explicit solutions for the fair strike price of discrete-time variance swaps constructed using the standard realized volatility definition. The resulting expression depends only on the contract maturity and on the first and second order partial derivatives of the coefficients in the one-step joint cumulant generating function. For the Gaussian case, which reduces to the standard HN-GARCH model, we provide a direct derivation for the variance swap price which coincides with the expression implied by the general methodology that we proposed for any affine GARCH model. Using the standard asymptotic conditions in Nelson (1990) we derive the continuous-time limit of the HN-GARCH model under the proposed variance dependent pricing kernel and we obtain the continuously sampled variance swap strike price using this limit. Finally, we establish the convergence of the GARCH-based discrete-time swap prices to this continuously sampled counterpart. We provide two numerical studies in which we analyze the term structure of the variance swaps under the HN-GARCH model implied by the historical asset returns on the S&P 500 index and we numerically test the theoretical convergence results. Interesting potential future research could be the derivation of analytical formulas for affine GARCH models or other volatility derivatives that exhibit a nonlinear dependence on the realized volatility, e.g. call options on the realized volatility.

Acknowledgements AB and JPO thank the hospitality of the Centre Interfacultaire Bernoulli of the Ecole Polytechnique Fédérale de Lausanne during the program "Stochastic Dynamical Models in Mathematical Finance, Econometrics, and Actuarial Sciences" that made possible the collaboration that lead to some of the results included in this paper. AB would like to thank NSERC for its continuing support.

A Appendix

A.1 Proof of Proposition 2.1

We let $l < k$ be arbitrarily chosen and evaluate the multi-step conditional joint c.g.f. of $\left(Y_{k\Delta}, h_{(k+1)\Delta}\right)$ using the law of iterated expectations:

$$\begin{aligned} C_{\left(Y_{k\Delta}, h_{(k+1)\Delta}\right)}\left(\phi, \psi \mid \mathcal{F}_{l\Delta}\right) &= \log \mathrm{E}^{P}\left[\exp\left(\phi Y_{k\Delta} + \psi h_{(k+1)\Delta}\right) \mid \mathcal{F}_{l\Delta}\right] \\ &= \log \mathrm{E}^{P}\left[\mathrm{E}^{P}\left[\exp\left(\phi Y_{k\Delta} + \psi h_{(k+1)\Delta}\right) \mid \mathcal{F}_{(l+1)\Delta}\right] \mid \mathcal{F}_{l\Delta}\right] \\ &= A\left(\phi, \psi; (l+1)\Delta, k\Delta\right) \\ &\quad + C_{\left(Y_{(l+1)\Delta}, h_{(l+2)\Delta}\right)}\left(\phi, B\left(\phi, \psi; (l+1)\Delta, k\Delta\right) \mid \mathcal{F}_{l\Delta}\right). \end{aligned}$$

Replacing the last term with its expression in (2.4), we obtain the required affine representation in (2.25) where the coefficients $A(\phi, \psi; l\Delta, k\Delta)$ and $B(\phi, \psi; l\Delta, k\Delta)$ satisfy the recursions (2.6)–(2.7). The terminal conditions $A(\phi, \psi; k\Delta, k\Delta) = 0$ and $B(\phi, \psi; k\Delta, k\Delta) = \psi$, for any real ϕ and ψ, follows immediately by replacing $l = k$ in the above equation. □

A.2 Proof of Proposition 2.2

We start by evaluating the joint c.g.f. of $\left(y_{k\Delta}, h_{(k+1)\Delta}\right)$ conditional on $\mathcal{F}_{(k-1)\Delta}$. We have:

$$\begin{aligned}
&C^{Q}_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\phi,\psi \mid \mathcal{F}_{(k-1)\Delta}\right)\\
&\quad=\log \mathrm{E}^{Q}\left[\exp\left(\phi y_{k\Delta}+\psi h_{(k+1)\Delta}\right) \mid \mathcal{F}_{(k-1)\Delta}\right]\\
&\quad=\log \mathrm{E}^{P}\Big[\exp\Big(\phi y_{k\Delta}+\psi h_{(k+1)\Delta}+\eta_1(\Delta)y_{k\Delta}+\eta_2(\Delta)h_{(k+1)\Delta}\\
&\qquad-\ C_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\eta_1(\Delta),\eta_2(\Delta) \mid \mathcal{F}_{(k-1)\Delta}\right)\Big) \mid \mathcal{F}_{(k-1)\Delta}\Big]\\
&=C_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\phi+\eta_1(\Delta),\psi+\eta_2(\Delta) \mid \mathcal{F}_{(k-1)\Delta}\right)\\
&\quad-C_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\eta_1(\Delta),\eta_2(\Delta) \mid \mathcal{F}_{(k-1)\Delta}\right).
\end{aligned}$$

Thus, we can write:

$$C^{Q}_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\phi,\psi \mid \mathcal{F}_{(k-1)\Delta}\right)=A^{Q}\left(\phi,\psi;(k-1)\Delta,k\Delta\right)+B^{Q}\left(\phi,\psi;(k-1)\Delta,k\Delta\right)h_{k\Delta},$$

where $A^{Q}\left(\phi,\psi;l\Delta,k\Delta\right)$ and $B^{Q}\left(\phi,\psi;l\Delta,k\Delta\right)$ are given in (2.10)–(2.11). The first no-arbitrage constraint is equivalent to the fact that Q is a well defined probability measure, which in our case is automatically verified since $C^{Q}_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(0,0 \mid \mathcal{F}_{(k-1)\Delta}\right)=0$. The second no-arbitrage constraint ensures that the discounted asset price is a martingale under Q, which is equivalent to $\mathrm{E}^{Q}\left[\exp\left(y_{k\Delta}\right) \mid \mathcal{F}_{(k-1)\Delta}\right]=\exp\left(r\Delta\right)$ for any $k=1,\cdots,nT$. Re-writing this using the joint c.g.f. specification, this is equivalent to $C^{Q}_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(1,0 \mid \mathcal{F}_{(k-1)\Delta}\right)=r\Delta$. Using (2.10)-(2.11), this condition implies that $\eta_1(\Delta)$ and $\eta_2(\Delta)$ satisfy the no-arbitrage constraints (2.12)–(2.13). □

A.3 Proof of Corollary 2.1

We notice that the first no-arbitrage constraint (2.12) is automatically satisfied for any $\eta_1(\Delta)$ and $\eta_2(\Delta 0)$, while the second constraint (2.13) leads to:

$$\lambda\Delta+\frac{\left((1+\eta_1(\Delta))\sqrt{\Delta}-2\alpha(\Delta)\gamma(\Delta)\eta_2(\Delta)\right)^2}{2\left(1-2\alpha(\Delta)\eta_2(\Delta)\right)}-\frac{\left(\eta_1(\Delta)\sqrt{\Delta}-2\alpha(\Delta)\gamma(\Delta)\eta_2(\Delta)\right)^2}{2\left(1-2\alpha(\Delta)\eta_2(\Delta)\right)}=0.$$

After some algebraic manipulations we can express the market price of equity risk as a function of the market price of variance risk as in (2.21).

Next, using the result from Proposition 2.2, we derive the conditional c.g.f. of the asset returns under Q:

$$\begin{aligned}
C^{Q}_{y_{k\Delta}}\left(\phi \mid \mathcal{F}_{(k-1)\Delta}\right)&=C^{Q}_{\left(y_{k\Delta},h_{(k+1)\Delta}\right)}\left(\phi,0 \mid \mathcal{F}_{(k-1)\Delta}\right)=A^{Q}\left(\phi,0;(k-1)\Delta,k\Delta\right)\\
&\quad+B^{Q}\left(\phi,0;(k-1)\Delta,k\Delta\right)h_{k\Delta}\\
&=\phi r\Delta+\left(\phi\lambda\Delta+\frac{\phi^2\Delta+2\phi\eta_1(\Delta)\Delta-4\phi\alpha(\Delta)\gamma(\Delta)\eta_2(\Delta)\sqrt{\Delta}}{2\left(1-2\alpha(\Delta)\eta_2(\Delta)\right)}\right)h_{k\Delta}\\
&=\phi\left(r-\frac{h_{k\Delta}}{2\left(1-2\alpha(\Delta)\eta_2(\Delta)\right)}\right)\Delta+\phi^2\frac{h_{k\Delta}}{2\left(1-2\alpha(\Delta)\eta_2(\Delta)\right)}\Delta.
\end{aligned}$$

If we denote $h^*_{k\Delta} = \dfrac{h_{k\Delta}}{2\,(1-2\alpha(\Delta)\eta_2(\Delta))}$, the above equation implies that $y_{k\Delta}\,|_{\mathcal{F}_{(k-1)\Delta}} \overset{Q}{\sim} \mathbf{N}\left(\left(r-\frac{h^*_{k\Delta}}{2}\right)\Delta, h^*_{k\Delta}\Delta\right)$. Thus, we can write the asset returns dynamics as in (2.22):

$$y_{k\Delta} = \left(r - \frac{h^*_{k\Delta}}{2}\right)\Delta + \sqrt{\Delta}\sqrt{h^*_{k\Delta}}\epsilon^*_{k\Delta}, \quad \epsilon^*_{k\Delta} \overset{Q}{\sim} \mathbf{N}(0,1).$$

From (2.14) and (2.22), we express the risk-neutral innovations in terms of the physical one as:

$$\epsilon_{k\Delta} = \frac{1}{\sqrt{\Delta}\sqrt{h_{k\Delta}}}\left(\sqrt{\Delta}\sqrt{h^*_{k\Delta}}\epsilon^*_{k\Delta} - \lambda h_{k\Delta}\Delta - \frac{h^*_{k\Delta}}{2}\Delta\right).$$

Multiplying the conditional variance equation (2.15) by $1/\,(1-2\alpha(\Delta)\eta_2(\Delta))$, and replacing the GARCH innovation process $\epsilon_{k\Delta}$ by the above expression, we obtain the risk-neutral dynamics from (2.23):

$$h^*_{(k+1)\Delta} = \omega^*(\Delta) + \beta(\Delta)h^*_{k\Delta} + \alpha^*(\Delta)\left(\epsilon^*_{k\Delta} - q^*(\Delta)\sqrt{h^*_{k\Delta}}\right)^2.$$

Note that the risk-neutral parameters are those illustrated in Corollary 2.1. □

A.4 Proof of Lemma 3.1

For any $k > l$, we evaluate the multi-step c.g.f. of $Y_{k\Delta} - Y_{(k-1)\Delta}$ given the filtration $\mathcal{F}_{l\Delta}$:

$$\begin{aligned}
C^Q_{Y_{k\Delta}-Y_{(k-1)\Delta}}(\phi \mid \mathcal{F}_{l\Delta}) &= \log \mathrm{E}^Q\left[\exp\left(\phi\left(Y_{k\Delta} - Y_{(k-1)\Delta}\right)\right) \mid \mathcal{F}_{l\Delta}\right] \\
&= \log \mathrm{E}^Q\left[\exp\left(-\phi Y_{(k-1)\Delta}\right)\mathrm{E}^Q\left[\exp\left(\phi Y_{k\Delta}\right) \mid \mathcal{F}_{(k-1)\Delta}\right] \mid \mathcal{F}_{l\Delta}\right] \\
&= \log \mathrm{E}^Q\left[\exp\left(-\phi Y_{(k-1)\Delta} + C^Q_{\left(Y_{k\Delta}, h^*_{(k+1)\Delta}\right)}\left(\phi, 0 \mid \mathcal{F}_{(k-1)\Delta}\right)\right) \mid \mathcal{F}_{l\Delta}\right] \\
&= \log \mathrm{E}^Q\left[\exp\left(A^*\left(\phi, 0; (k-1)\Delta, k\Delta\right) + B^*\left(\phi, 0; (k-1)\Delta, k\Delta\right)h^*_{k\Delta}\right) \mid \mathcal{F}_{l\Delta}\right] \\
&= A^*\left(\phi, 0; (k-1)\Delta, k\Delta\right) + C^Q_{\left(Y_{(k-1)\Delta}, h^*_{k\Delta}\right)}\left(0, B^*\left(\phi, 0; (k-1)\Delta, k\Delta\right) \mid \mathcal{F}_{l\Delta}\right).
\end{aligned}$$

Thus, we have the following affine representation:

$$\begin{aligned}
C^Q_{Y_{k\Delta}-Y_{(k-1)\Delta}}(\phi \mid \mathcal{F}_{l\Delta}) = {} & A^*\left(\phi, 0; (k-1)\Delta, k\Delta\right) \\
& + A^*\left(0, B^*\left(\phi, 0; (k-1)\Delta, k\Delta\right); l\Delta, (k-1)\Delta\right) \\
& + B^*\left(0, B^*\left(\phi, 0; (k-1)\Delta, k\Delta\right); l\Delta, (k-1)\Delta\right)h^*_{(l+1)\Delta}.
\end{aligned}$$

□

A.5 Proof of Proposition 3.1

Using (3.3) and the notations from (3.4), we express the first and second order cumulants of the one-step return process $y_{k\Delta}$ in the following way:

$$\kappa^{(1)}_{y_{k\Delta}} = A^*_{10} + B^*_{10}\left(\frac{\partial A^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi}\,|_{\psi=0} + \frac{\partial B^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi}\,|_{\psi=0}\; h^*_{\Delta}\right), \tag{A.1}$$

$$\begin{aligned}\kappa^{(2)}_{y_{k\Delta}} &= A^*_{20} + B^*_{20}\left(\frac{\partial A^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi}\,|_{\psi=0} + \frac{\partial B^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi}\,|_{\psi=0}\; h^*_{\Delta}\right)\\ &\quad + B^{*2}_{10}\left(\frac{\partial^2 A^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi^2}\,|_{\psi=0} + \frac{\partial^2 B^*\left(0,\psi;0,(k-1)\Delta\right)}{\partial\psi^2}\,|_{\psi=0}\; h^*_{\Delta}\right).\end{aligned} \tag{A.2}$$

We have used the fact that $B^*\left(0,0;l\Delta,k\Delta\right)=0$, for any $l<k$. Thus, in order to evaluate the above expressions, we only need to compute the first and second order derivatives of the multi-step coefficients $A^*\left(0,\psi;0,(k-1)\Delta\right)$ and $B^*\left(0,\psi;0,(k-1)\Delta\right)$. This is illustrated below more generally for any time points $l\Delta$ and $k\Delta$ with $l<k$.

First, following (2.6)–(2.7), we recall the risk-neutral recursions:

$$\begin{aligned}A^*\left(0,\psi;l\Delta,k\Delta\right) &= A^*\left(0,\psi;(l+1)\Delta,k\Delta\right)\\ &\quad + A^*\left(0,B\left(0,\psi;(l+1)\Delta,k\Delta\right);l\Delta,(l+1)\Delta\right). \qquad\qquad (A.3)\\ B^*\left(0,\psi;l\Delta,k\Delta\right) &= B^*\left(0,B^*\left(0,\psi;(l+1)\Delta,k\Delta\right);l\Delta,(l+1)\Delta\right), \qquad\qquad (A.4)\end{aligned}$$

with the terminal conditions $A\left(0,\psi;k\Delta,k\Delta\right)=0$ and $B\left(0,\psi;k\Delta,k\Delta\right)=\psi$, for any real ψ.

Taking the first derivative of (A.4) with respect to ψ and we evaluate it at $\psi=0$, we obtain:

$$\frac{\partial B^*\left(0,\psi;l\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0} = B^*_{01}\,\frac{\partial B^*\left(0,\psi;(l+1)\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0}\,. \tag{A.5}$$

Solving (A.5) subject to the terminal constraint $\frac{\partial B^*\left(0,\psi;k\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0}=1$, we obtain the following solution:

$$\frac{\partial B^*\left(0,\psi;l\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0} = B^{*k-l}_{01}. \tag{A.6}$$

Similarly, taking the first derivative of (A.3) with respect to ψ and evaluating it at $\psi=0$, we have:

$$\begin{aligned}\frac{\partial A^*\left(0,\psi;l\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0} &= \frac{\partial A^*\left(0,\psi;(l+1)\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0}\\ &\quad + A^*_{01}\,\frac{\partial B^*\left(0,\psi;(l+1)\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0}\,.\end{aligned} \tag{A.7}$$

Solving (A.7) subject to the terminal constraint $\frac{\partial A^*\left(0,\psi;k\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0}=0$, we obtain the following solution:

$$\frac{\partial A^*\left(0,\psi;l\Delta,k\Delta\right)}{\partial\psi}\,|_{\psi=0} = A^*_{01}\,\frac{1-B^{*k-l}_{01}}{1-B^*_{01}}. \tag{A.8}$$

We proceed in a similar fashion for the second order derivatives. From equation (A.4), we have:

$$\frac{\partial^2 B^*(0,\psi;l\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} = B^*_{01}\frac{\partial^2 B^*(0,\psi;(l+1)\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} + B^*_{02}\left(\frac{\partial B^*(0,\psi;(l+1)\Delta,k\Delta)}{\partial\psi}|_{\psi=0}\right)^2. \quad (A.9)$$

Solving (A.9) subject to the terminal constraint $\frac{\partial^2 B^*(0,\psi;k\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0}=0$, we obtain the following solution:

$$\frac{\partial^2 B^*(0,\psi;l\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0}= B^*_{02}B^{*k-l-1}_{01}\frac{1-B^{*k-l}_{01}}{1-B^*_{01}}. \quad (A.10)$$

Finally from equation (A.3) we have:

$$\frac{\partial^2 A^*(0,\psi;l\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} = \frac{\partial^2 A^*(0,\psi;(l+1)\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} + A^*_{01}\frac{\partial^2 B^*(0,\psi;(l+1)\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} + A^*_{02}\left(\frac{\partial B^*(0,\psi;(l+1)\Delta,k\Delta)}{\partial\psi}|_{\psi=0}\right)^2 \quad (A.11)$$

Equation (A.11), together with its terminal condition $\frac{\partial^2 A^*(0,\psi;k\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0}=0$, leads to the following solution:

$$\frac{\partial^2 B^*(0,\psi;l\Delta,k\Delta)}{\partial\psi^2}|_{\psi=0} = A^*_{02}\frac{1-B^{*2(k-l)}_{01}}{1-B^{*2}_{01}} + \frac{A^*_{01}B^*_{02}}{B^*_{01}(1-B^*_{01})}\left(\frac{1-B^{*k-l}_{01}}{1-B^*_{01}} - \frac{1-B^{*2(k-l)}_{01}}{1-B^{*2}_{01}}\right). \quad (A.12)$$

Replacing (A.6), (A.8), (A.10) and (A.12) with $l \to 0$ and $k \to k-1$ into the cumulant expressions from (A.1)–(A.2), and after some algebraic manipulations, we can express the variance swap strike price as:

$$\begin{aligned} K(0,T,nT) &= \mathrm{E}^Q\left[\frac{1}{T}\sum_{k=1}^{nT}\left(Y_{k\Delta}-Y_{(k-1)\Delta}\right)^2\right] = \frac{1}{T}\sum_{k=1}^{nT}\left(\left(\kappa^{(1)}_{y_{k\Delta}}\right)^2+\kappa^{(2)}_{y_{k\Delta}}\right)\\ &= nF_1 + \frac{F_2}{T}\frac{1-B^{*nT}_{01}}{1-B^*_{01}} + \frac{F_3}{T}\frac{1-B^{*2nT}_{01}}{1-B^{*2}_{01}}\\ &+ \left(\frac{F_4}{T}\frac{1-B^{*nT}_{01}}{1-B^*_{01}} + \frac{F_5}{T}\frac{1-B^{*2nT}_{01}}{1-B^{*2}_{01}}\right)h^*_\Delta\\ &+ \frac{F_6}{T}\frac{1-B^{*2nT}_{01}}{1-B^{*2}_{01}}h^{*2}_\Delta. \end{aligned}$$

Here the factors F_1 through F_6 are those defined in (3.6)–(3.11). □

A.6 Proof of Corollary 3.1

The risk-neutral dynamics of the HN model in Corollary 2.1 imply the following one-step ahead coefficients in the affine representation:

$$
\begin{aligned}
A^*(\phi, \psi; l\Delta, (l+1)\Delta) &= \phi r \Delta + \psi \omega^*(\Delta) - \frac{1}{2}\log\left(1 - 2\psi\alpha^*(\Delta)\right), \\
B^*(\phi, \psi; l\Delta, (l+1)\Delta) &= -\frac{\phi\Delta}{2} + \psi\left(\alpha^*(\Delta)(q^*(\Delta))^2 + \beta(\Delta)\right) \\
&\quad + \frac{\left(\phi\sqrt{\Delta} - 2\psi\alpha^*(\Delta)q^*(\Delta)\right)^2}{2\left(1 - 2\psi\alpha^*(\Delta)\right)},
\end{aligned}
$$

for any $l = 0, \ldots, nT - 1$. Taking the first and second order partial derivatives of the above with respect to ϕ and ψ at $(0, 0)$, and using the notations from (A.13), we easily obtain the expressions from (3.12). The variance swap price expression in the Heston and Nandi case follows by replacing these coefficients into the general formula (3.5). □

Alternative derivation of the variance swap price for the HN model First, we introduce several useful notations:

$$
\begin{aligned}
m^*(\Delta) &= \frac{\omega^*(\Delta) + \alpha^*(\Delta)}{1 - d^*(\Delta)}; \quad b^*(\Delta) = 2\left(\alpha^*(\Delta)\right)^2 + \left(m^*(\Delta)\right)^2\left(1 - d^*(\Delta)\right)^2; \\
c^*(\Delta) &= 4\left(\alpha^*(\Delta)\right)^2\left(q^*(\Delta)\right)^2 + 2m^*(\Delta)d^*(\Delta)\left(1 - d^*(\Delta)\right).
\end{aligned}
\tag{A.13}
$$

From (2.22), we have:

$$
\mathrm{E}^Q\left[\left(Y_{k\Delta}^{(n)} - Y_{(k-1)\Delta}^{(n)}\right)^2\right] = r^2\Delta^2 + \Delta(1 - r\Delta)\mathrm{E}^Q\left[h_{k\Delta}^*\right] + \frac{\Delta^2}{4}\mathrm{E}^Q\left[(h_{k\Delta}^*)^2\right].
\tag{A.14}
$$

Thus, our goal reduces to computing the first two unconditional moments of $h_{k\Delta}^*$. Taking expectations on both sides of (2.23), we obtain:

$$
\mathrm{E}^Q\left[h_{(k+1)\Delta}^*\right] = \omega^*(\Delta) + \alpha^*(\Delta) + \left(\alpha^*(\Delta)q^*(\Delta)^2 + \beta(\Delta)\right)\mathrm{E}^Q\left[h_{k\Delta}^*\right].
\tag{A.15}
$$

Solving (A.15) recursively and using the notations from (A.13), we get:

$$
\mathrm{E}^Q[h_{k\Delta}^*] = (d^*(\Delta))^{k-1}(h_\Delta^* - m^*(\Delta)) + m^*(\Delta).
\tag{A.16}
$$

Next, we square both sides of (2.23) and we have:

$$
\begin{aligned}
h_{(k+1)\Delta}^{*2} &= (\omega^*(\Delta) + \beta(\Delta)h_{k\Delta}^*)^2 + (\alpha^*(\Delta))^2(\epsilon_{k\Delta}^* - q^*(\Delta)\sqrt{h_{k\Delta}^*})^4 \\
&\quad + 2(\omega^*(\Delta) + \beta(\Delta)h_{k\Delta}^*)\alpha^*(\Delta)(\epsilon_{k\Delta}^* - q^*(\Delta)\sqrt{h_{k\Delta}^*})^2.
\end{aligned}
$$

Taking the expectation on both sides of the above we obtain the following recursion:

$$
\begin{aligned}
\mathrm{E}^Q[h_{(k+1)\Delta}^{*2}] &= (\omega^*(\Delta))^2 + 3(\alpha^*(\Delta))^2 + 2\alpha^*(\Delta)\omega^*(\Delta) \\
&\quad + \left(2\omega^*(\Delta)\beta(\Delta) + 6(\alpha^*(\Delta))^2(q^*(\Delta))^2\right. \\
&\quad \left. + 2\alpha^*(\Delta)(\beta(\Delta) + \omega^*(\Delta)(q^*(\Delta))^2)\right)\mathrm{E}^Q\left[h_{k\Delta}^*\right] \\
&\quad + \left(\beta(\Delta) + \alpha^*(\Delta)(q^*(\Delta))^2\right)^2\mathrm{E}^Q\left[h_{k\Delta}^{*2}\right].
\end{aligned}
\tag{A.17}
$$

Denoting the sum of the first two terms of the right hand side of (A.17) by $I_k(\Delta)$, and using the notations from (A.13), we obtain a simplified recursion for the unconditional second moment:

$$\mathrm{E}^Q[h^{*2}_{(k+1)\Delta}] = I_k(\Delta) + \left(d^*(\Delta)\right)^2 \mathrm{E}^Q\left[h^{*2}_{k\Delta}\right]. \tag{A.18}$$

Here, the intermediate term $I_k(\Delta)$ is given by:

$$I_k(\Delta) = b^*(\Delta) + c^*(\Delta)\mathrm{E}^Q\left[h^*_{k\Delta}\right] = b^*(\Delta) + c^*(\Delta)m^*(\Delta) + c^*(\Delta)\left(d^*(\Delta)\right)^{k-1} \times (h^*_\Delta - m^*(\Delta)).$$

Solving (A.18) recursively we have:

$$\begin{aligned}
\mathrm{E}^Q[h^{*2}_{k\Delta}] &= \sum_{j=1}^{k-1}\left(d^*(\Delta)\right)^{2(j-1)} I_{k-j}(\Delta) + \left(d^*(\Delta)\right)^{2(k-1)} h^{*2}_\Delta \\
&= (b^*(\Delta) + c^*(\Delta)m^*(\Delta))\frac{1-(d^*(\Delta))^{2(k-1)}}{1-(d^*(\Delta))^2} \\
&\quad - c^*(\Delta)m^*(\Delta)\frac{(d^*(\Delta))^{k-2} - (d^*(\Delta))^{2k-3}}{1-d^*(\Delta)} \\
&\quad + c^*(\Delta)\frac{(d^*(\Delta))^{k-2} - (d^*(\Delta))^{2k-3}}{1-d^*(\Delta)} h^*_\Delta + \left(d^*(\Delta)\right)^{2(k-1)} h^{*2}_\Delta.
\end{aligned} \tag{A.19}$$

Replacing (A.16) and (A.19) into (A.14), we obtain the following quadratic expression (as a function of h^*_Δ):

$$\begin{aligned}
\mathrm{E}^Q\left[\left(Y^{(n)}_{k\Delta} - Y^{(n)}_{(k-1)\Delta}\right)^2\right] &= r^2\Delta^2 + \Delta(1-r\Delta)m^*(\Delta)\left(1 - \left((d^*(\Delta)\right)^{k-1}\right) \\
&+ \frac{\Delta^2 b^*(\Delta)}{4}\frac{1-(d^*(\Delta))^{2(k-1)}}{1-(d^*(\Delta))^2} \\
&+ \frac{\Delta^2 c^*(\Delta)m^*(\Delta)}{4}\left(\frac{1-(d^*(\Delta))^{2(k-1)}}{1-(d^*(\Delta))^2} - \frac{(d^*(\Delta))^{k-2} - (d^*(\Delta))^{2k-3}}{1-d^*(\Delta)}\right) \\
&+ \left(\Delta(1-r\Delta)\left(d^*(\Delta)\right)^{k-1} + \frac{\Delta^2 c^*(\Delta)}{4}\frac{(d^*(\Delta))^{k-2} - (d^*(\Delta))^{2k-3}}{1-d^*(\Delta)}\right) h^*_\Delta \\
&+ \frac{\Delta^2}{4}\left(d^*(\Delta)\right)^{2(k-1)} h^{*2}_\Delta.
\end{aligned}$$

It now follows that the variance swap price is given by:

$$K^{(n)}_{(0,T,nT)} = \frac{1}{T}\sum_{k=1}^{nT}\mathrm{E}^Q\left[\left(Y^{(n)}_{k\Delta} - Y^{(n)}_{(k-1)\Delta}\right)^2\right] = \xi^*_0(\Delta) + \xi^*_1(\Delta)h^*_\Delta + \xi^*_2(\Delta)h^{*2}_\Delta, \tag{A.20}$$

where the coefficients $\xi_0^*(\Delta)$, $\xi_1^*(\Delta)$ and $\xi_2^*(\Delta)$ are given by:

$$\begin{aligned}\xi_0^*(\Delta) &= r^2\Delta + (1-r\Delta)m^*(\Delta) - \frac{\Delta(1-r\Delta)m^*(\Delta)}{T}\frac{1-(d^*(\Delta))^{nT}}{1-d^*(\Delta)}\\ &\quad + \frac{\Delta\left(b^*(\Delta)+c^*(\Delta)m^*(\Delta)\right)}{4\left(1-(d^*(\Delta))^2\right)}\left(1-\frac{\Delta}{T}\frac{1-(d^*(\Delta))^{2nT}}{1-(d^*(\Delta))^2}\right)\\ &\quad - \frac{\Delta^2 c^*(\Delta)m^*(\Delta)}{4Td^*(\Delta)\left(1-d^*(\Delta)\right)}\left(\frac{1-(d^*(\Delta))^{nT}}{1-d^*(\Delta)} - \frac{1-(d^*(\Delta))^{2nT}}{1-(d^*(\Delta))^2}\right),\\ \xi_1^*(\Delta) &= \frac{\Delta(1-r\Delta)}{T}\frac{1-(d^*(\Delta))^{nT}}{1-d^*(\Delta)}\\ &\quad + \frac{\Delta^2 c^*(\Delta)}{4Td^*(\Delta)\left(1-d^*(\Delta)\right)}\left(\frac{1-(d^*(\Delta))^{nT}}{1-d^*(\Delta)} - \frac{1-(d^*(\Delta))^{2nT}}{1-(d^*(\Delta))^2}\right),\\ \xi_2^*(\Delta) &= \frac{\Delta^2}{4T}\frac{1-(d^*(\Delta))^{2nT}}{1-(d^*(\Delta))^2}.\end{aligned}$$

It is not difficult, although tedious, to show that the above formula and the more general variance swap price in (3.5) coincide in the case of the HN model. □

A.7 Proof of Proposition 4.1

We use the weak convergence theorem of Markov processes to diffusions (see e.g. Theorem 12.1 on page 314 of Francq and Zakoian (2011)) in order to compute the drift and the diffusion coefficients of the following risk-neutral limiting process:

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t)dt + \mathbf{b}(\mathbf{X}_t)d\mathbf{W}_t^*. \tag{A.21}$$

Here $\mathbf{X}_t = (Y_t, h_t)^T$ and $\mathbf{W}_t^* = \left(W_{1t}^*, W_{2t}^*\right)^T$, with W_{1t}^* and W_{2t}^* being two independent standard Brownian motions under Q.

Using the relationships between moments and cumulants, together with the coefficients derived in Corollary 3.1, we evaluate the first and second limiting moments of $Y_{k\Delta}$ and $h^*_{(k+1)\Delta}$ under Q, conditional on the filtration $\mathcal{F}^h_{(k-1)\Delta} := \mathcal{F}_{(k-1)\Delta} \bigcup \{h_{k\Delta} = h_t\}$. We have:

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}^*\left[\Delta Y_{k\Delta}\middle|\mathcal{F}^h_{(k-1)\Delta}\right] &= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(A_{10}^* + B_{10}^* h_{k\Delta}^*\right) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(r\Delta - \frac{\Delta}{2}h^*_{k\Delta}\right)\\ &= r - \frac{1}{2}h_t.\\ \lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}^*\left[\Delta h^*_{(k+1)\Delta}\middle|\mathcal{F}^h_{(k-1)\Delta}\right] &= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(A_{01}^* + \left(B_{01}^*-1\right)h^*_{k\Delta}\right)\\ &= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(\omega^*(\Delta)+\alpha^*(\Delta) - \left(1-d^*(\Delta)\right)h^*_{k\Delta}\right)\\ &= \omega + \alpha - \left(k - \sigma\left(\lambda + \frac{1}{2}\right)\right)h_t = \kappa^*(\theta^* - h_t).\end{aligned}$$

Thus, the the drift term of the limiting diffusion is given by:

$$\mathbf{a}(Y_t, h_t) = \begin{pmatrix} r - \frac{h_t}{2} \\ \kappa^*(\theta^* - h_t)\end{pmatrix}$$

The second moment computations are illustrated below:

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{Var}^*\left[\Delta Y_{k\Delta}\middle|\mathcal{F}^h_{(k-1)\Delta}\right] = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(A^*_{20}+B^*_{20}h^*_{k\Delta}\right)$$
$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(0+\Delta h^*_{k\Delta}\right) = h_t.$$
$$\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{Var}^*\left[\Delta h_{(k+1)\Delta}\middle|\mathcal{F}^h_{(k-1)\Delta}\right] = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(A^*_{02}+B^*_{02}h^*_{k\Delta}\right)$$
$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(2\left(\alpha^*(\Delta)\right)^2+4\left(\alpha^*(\Delta)\right)^2\left(q^*(\Delta)\right)^2 h^*_{k\Delta}\right)$$
$$= \sigma^2 h_t.$$
$$\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{Cov}^*\left[\Delta Y_{k\Delta},\Delta h_{(k+1)\Delta}\middle|\mathcal{F}^h_{(k-1)\Delta}\right] = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(A^*_{11}+B^*_{11}h^*_{k\Delta}\right)$$
$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(0-2\alpha^*(\Delta)q^*(\Delta)h^*_{k\Delta}\right) = -\sigma h_t.$$

Thus, the risk-neutral second moment matrix is given by:

$$\Sigma(Y_t,h_t)\Sigma^T(Y_t,h_t) = \begin{pmatrix} h_t & -\sigma h_t \\ -\sigma h_t & \sigma^2 h_t \end{pmatrix}.$$

Using a Cholesky decomposition of $\Sigma(Y_t,h_t)$, leads to the diffusion coefficient of the diffusion limit:

$$\Sigma(Y_t,h_t) = \begin{pmatrix} \sqrt{h_t} & 0 \\ -\sigma\sqrt{h_t} & 0 \end{pmatrix}. \quad \text{(A.22)}$$

Replacing the resulting drift and diffusion coefficients into (A.21) we obtain the dynamics from (4.5)–(4.6). □

A.8 Proof of Proposition 4.2

Proof (i) See Broadie and Jain (2008).

(ii) In order to compute the limits of the GARCH-based variance swap prices, we need to establish some useful results which shall be used throughout the proof. First, we notice that as $\Delta \to 0$, or equivalently $n \to 0$, the risk-neutral persistence and the quantities introduced in (A.13) have the following behavior:

$$d^*(\Delta)\to 1, \quad m^*(\Delta)\to\theta^*, \quad b^*(\Delta)\to 0, \quad c^*(\Delta)\to 0. \quad \text{(A.23)}$$

Next, it is easy to check that:

$$\lim_{\Delta\to 0}\frac{1-d^*(\Delta)}{\Delta} = \kappa^*, \quad \lim_{\Delta\to 0}\frac{1-(d^*(\Delta))^2}{\Delta} = 2\kappa^*, \quad \lim_{\Delta\to 0}\left(d^*(\Delta)\right)^n = e^{-\kappa^*}. \quad \text{(A.24)}$$

We now compute the limit of each coefficient from the quadratic representation in (A.20). We have: Therefore, based on the (A.23) and (A.24), we find that:

$$\lim_{\Delta\to 0}\xi^*_0(\Delta) = \theta^* - \frac{\theta^*\left(1-e^{-\kappa^* T}\right)}{\kappa^* T}, \quad \lim_{\Delta\to 0}\xi^*_1(\Delta) = \frac{1-e^{-\kappa^* T}}{\kappa^* T}, \quad \lim_{\Delta\to 0}\xi^*_2(\Delta) = 0. \quad \text{(A.25)}$$

Replacing (A.25) into the limit of equation (A.20) we find that $\lim_{\Delta\to 0} K(0,T,nT) = \bar{K}(0,T)$, where $\bar{K}(0,T)$ is given in (4.8). The same limit can be obtained if we directly compute the limits of all quantities in Proposition 3.1. □

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